ELEMENTS
OF
GEOMETRY;
WITH THEIR
Application to the Menſuration
of SUPERFICIES and SOLIDS,
TO THE
Determination of the MAXIMA and MINIMA of Geometrical Quantities,
AND TO THE
Conſtruction of a great Variety of GEOMETRICAL PROBLEMS.

By *THOMAS SIMPSON*, F. R. S.
And Member of the Royal Academy of Sciences at STOCKHOLM.

The SECOND EDITION,
With large ALTERATIONS and ADDITIONS.

LONDON,
Printed for J. NOURSE at the *Lamb* oppoſite *Katherine-Street* in the *Strand*. M.DCC.LX.

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GEOMETRY:

OR

ELEMENTS

OF

Geometrical Geometries

DETERMINATION OF THE MAXIMA AND MINIMA

AND TO THE

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WITH THEIR

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TO

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TO THE HONOURABLE

Charles Frederick, Esq;

Surveyor-General of his MAJESTY'S ORDNANCE, *&c. &c. &c.*

HONOURABLE SIR,

THE subject of the sheets which I here beg leave to lay before You, is of so much consequence to mankind, as justly to claim the regard and sanction of the Great. Geometry is, not only a most accurate, but a very extensive science, whose application and great utility, as well in the arts of peace as of war, are well known to You.

But though this work, if the manner in which it is executed be correspondent to the importance of the subject; may not want sufficient merit to render it worthy of the approbation of a Gentleman, who, amidst a multiplicity of publick employments, preserves an undiminish'd ardor for the sciences

and a knowledge of the works of art and nature; yet I have, Sir, ſtill farther motives for this addreſs: Your great influence and zeal to promote the good of an inſtitution under which I am placed; and the favours that I have received at your hands, make me earneſt to embrace this opportunity of teſtifying publickly, that I am,

HONOURABLE SIR,

With great reſpect,

Your much obliged,

and moſt obedient

humble ſervant,

Royal Academy,
March 3, 1760.

Thomas Simpſon.

PREFACE.

MY design in writing upon the subject of Geometry, was to open an easy way for young beginners to arrive at a proficiency in that useful science; without either being obliged to go thro' a number of unnecessary propositions, or having recourse to the ungeometrical *methods of demonstration, that abound in most modern compositions of this nature.*

The difficulty of the undertaking, I was not unapprised of; and objections occurred that were not easy to be removed: Nevertheless, I have grounds to hope, from the reception my first attempt has met with, that my endeavours have not been entirely unsuccessful. No pains have, indeed, been spared to render the work useful: And I flatter myself that the spirit and rigour of demonstration, so essential to the subject, are also tolerably well preserved; though I have not been so intent to guard against the attack of Criticks, *as to lose sight of my main design of furnishing a plain, easy institution for learners: Yet I have strong hopes, that there will not be found in these sheets, any inaccuracies, or oversights, that are absolutely unpardonable. To expect a faultless piece is impossible: And I well know, that the most elaborate and best-approved systems of Geometry extant, are not without many imperfections. But, were the smallest imperfection to be a real fault, my ambition would rather be, to shew some degree of judgment, by avoiding a multitude of such faults, than by exposing and magnifying the flaws of other writers. It is more easy to see a fault, than to avoid one: And those men who are the most sanguine to distinguish themselves at the expence of others, are*

generally observed to stand in need of greater indulgencies, than even the persons whom they unmercifully attack. But I shall put an end to this digression by pointing out one objection, that may be brought against this work; which is, that in demonstrations admitting of several cases, the most easy ones are sometimes omitted; and that the converse of some propositions is not at all demonstrated. But this, I conceive, will be found a real advantage to the learner; without which, it would have been impossible to have comprised the Elements in the compass they now take up. Besides, the greatest part of the demonstrations omitted being such as may be inferred from those given, by means of Axioms only; they may, therefore, be easily supplied by any reader, should they happen to become necessary, which I have scarce ever found to be the case. But, even allowing this to be a defect, it is abundantly compensated by the extensive application given in the three last sections; which is infinitely more useful, in itself, and more necessary to the forming an able Geometrician, *than any thing of the kind we have been speaking of.*

In this, second, edition (which is, in a manner, a new work) many considerable alterations and additions have been made. The order of some of the first propositions is changed: And some difficult propositions in the second book are rendered more plain. In the fourth book several new Theorems on proportions are added. The solid Geometry is now connected with the plane, *and is demonstrated with the same accuracy. The mensuration of Superficies and Solids is also more explicitly handled; and the demonstration of the several rules is here established on a better foundation, than even in authors who have wrote professedly on the subject. The* Maxima *and* Minima, *and the construction of Geometrical Problems, are likewise considerably extended and improved. And, at the end,*

Notes geometrical and critical, very useful to improve the judgment of young students, are now added.

But, whilst I am talking of improvements and matters of criticism, I am called upon to answer to a charge, which, should it appear to deserve credit, would indeed leave me but little room to pass myself upon the world for a judge in these matters. As the gentleman by whom I stand accused, is known to the world by his holding one of the most considerable mathematical posts in the kingdom; I shall, in order to do all due honour to the manner and importance of his writing, give you his own words.

" *There has lately been published a book under the*
" *title of Elements of Plane Geometry, designed for*
" *the use of schools, which is an incorrect copy of the*
" *first eight sections of this work, lent the pretended*
" *author on a particular occasion, and printed in a*
" *spurious manner, without my knowledge or consent;*
" *an action too scandalous for any man of honour to*
" *be guilty of. The Editor imagined, I suppose, that*
" *the changing some propositions, and mangling the*
" *demonstrations of others, was a sufficient disguise*
" *to make it pass for his own performance; but how*
" *far this will justify such a piece of pyracy, must be*
" *left to the judgment of the publick.*"

Were I to attempt to describe the ideas excited in my mind by the singular modesty *of this important and solemn appeal to the publick, I should be at a loss for fit words to express them, without transgressing the bounds of decency. But I hope that I have not deserved so ill of the publick, to be thought capable of acting so very humble a part, as that of copying from* this *author, and of mangling* his *demonstrations, in order to make them pass for* my own.—*That a manuscript of his (containing between* 20 *and* 30 *of the*

principal

principal Theorems in Geometry, extremely ill digested) came into my hands, is indeed true; but it was not lent me, *but forced upon me, by himself (the very first night after my removal to* Woolwich) *in virtue of an* article *in the original rules and instructions for the Academy; whereby it is ordered, that the second master shall teach Geometry under the direction of the first master. But this well-intended* article, *which has been made subservient to the purposes of ignorant tyranny, and daring calumny, has since, in consequence of a publick examination, been annulled by an express order of the Master-General of the Ordnance.—I could mention some particulars, supported by good authority, that occurred in the course of that examination, which would but ill agree with the importance he assumes in his confident accusation; but I do not think it worth while: This Gentleman has, himself, by his different publications, so well convinced the world of his abilities, as to render any farther comment on that head intirely unnecessary and ineffectual.*

ERRATA.

Page 19. l. 4. for *x*, read *m*; p. 47. l. 23. for CAE, *r.* DAE; p. 48 l. 31. *r.* semi-diameters; p. 58. for Corollary 2. *r.* Corollary; p. 59. l. 12. *dele* the comma; p. 70. l. 15. for DE, *r.* DF; l. 31. for greater, *r.* greater than; p. 71. l. last, for PT, *r.* QT; p. 101. l. 6. for *y*, r. *z*; p. 110. in the quotation [l], for 19, *r.* 18; p. 124. l. 8. *dele* KF; p. 127. *l.* 4. *dele* the whole parenthesis; and in the quotation [c], *r.* 27. 4. p. 135. l. 25. after BD, *r.* as well as AD; p. 140. l. 14. for GEA, *r.* GEB; p. 154. beg. of l. 16. for CE, *r.* CD; p. 174. l. 25. for OHI, *r.* OHK; l. 34. *r.* annulus; p. 187. l. 21. *r.* frustum's height; p. 221. l. 20. for AG, *r.* AF; p. 226. l. prec. last, for exceeds, *r.* is less than; p. 248. l. 12. for less, *r.* greater; p. 267. l. 27. dele *not*; p. 271. l. 32. for which &c. *r.* when taken less.

ADVER-

ADVERTISEMENT.

AS in every work of this nature, designed to contain whatever may be moſt requiſite to the forming of a regular and complete ſyſtem of Geometry, a number of propoſitions muſt neceſſarily have a place, whoſe chief uſe and application lie in the higher branches of the Mathematicks; and there being many perſons, particularly young gentlemen in publick ſchools, who want to learn ſo much Geometry only, as is neceſſary to give them a proper introduction into the practical and moſt common applications thereof; ſuch as Menſuration, Trigonometry, Navigation, Fortification, Perſpective, *&c.* For theſe reaſons, I thought that it might be of ſervice, to point out to ſuch Readers, what propoſitions in theſe Elements may be omitted, as leaſt uſeful to them; without either hurting the connection, or taking away from the evidence of the other demonſtrations. The numbers of theſe propoſitions, in the ſeveral books, are as follow.

In Book I. the 6, 17, 19, 21, 22, 23, and 29th.

In Book II. the 4, 5, 10, 11, 12, 13th, and the 2d Corol. to the 9th.

In Book III. the 4, 5, 6, 7, 8, 9, 15, 18, 19, 20, 25, 26, 27, and 28th.

In Book IV. the 4, 5, 6, 9, 11, 13, 16, 17, 20, 21, 22, 23, 25, 26, 27, 28, and 29th.

In

In Book V. the 1, 2, 16, 17, 18, 19, 20, 25, 26, 28, and 31ſt.

In Book VI. the two or three firſt propoſitions only, need be read; except by thoſe who are concerned in ſurveying and dividing of lands; to whom the whole book will be highly uſeful.

Alſo, with regard to the ſeventh book, if *Perſpective* be the only application in view (which I have known frequently to be the caſe) the 1ſt, 2d, 4th, and 12th propoſitions may ſuffice. But if a more general idea of the properties of interſecting planes ſhould be required, ſuch as is neceſſary in the doctrine of ſolids and ſpheric geometry; then all the propoſitions, to the 12th, ought to be taken.

The 17th, 19th, 20th, 21ſt, 22d, and 23d propoſitions of this ſeventh book ſhould alſo be read by thoſe who would be able to find the content and proportion of ſolid bodies; as ſhould, likewiſe, the whole eighth book; except, perhaps, the firſt and ninth propoſitions, together with the three firſt lemmas; which may be thought too plain, by thoſe who are not very ſolicitous about geometrical rigour, to need a demonſtration.

An INDEX *or* TABLE *referring to the places in these Elements, where all the moſt material propoſitions in the firſt ſix, and in the eleventh and twelfth books of* Euclid, *are demonſtrated.*

Euclid.	Theſe El.
Book I.	P. B.
Prop. 5	12. 1
6	18. 1
8	14. 1
9	5. 5
10	6. 5
11	3. 5
12	4. 5
13	1. 1
15	3. 1
18 } 19 }	13. 1
21	23. 1
22	8. 5
23	7. 5
24	21. 1
26	15. 1
27 } 28 }	8. 1
29	7. 1
31	9. 5
32	9. 1 { 10. 1
33	26. 1
34	24. 1
35	2. 2
36 } 37 } 38 } 41 }	Cor. to 2. 2
42	6. 6
43	3. 2
44	6. 6
45	7. 6
46	10. 5
47	8. 2
B II.	P. B.
Pr. 1	5. 2
4	6. 2
5	7. 2

Euclid.	Theſe El.
B. II.	P. B.
P. 11	19. 5
12 } 13 }	10. 2
14	9. 6
B. III.	P. B.
Pr. 3	2. 3
7 } 8 }	5. 3
11	8. 3
12	7. 3
14	3. 3
15	4. 3
17	21. 5
20	10. 3
21	11. 3
22	17. 3
25	18. 3
31	16. 3
32	14. 3
33	22. 5
34	25. 5
35	21. 3
36 } 37 }	22. 3
B. IV.	P. B.
Pr. 2	25. 5
3	26. 5
4	24. 5
5	23. 5
10 } 11 }	28. 5
12	30. 5
15	29. 5
B. V.	P. B.
4	4. 4
12	6. 4
15	1. 4

Euclid.	Theſe El.
B. V.	P. B.
P. 16	2. 4
17 } 18 } 19 }	3. 4
22	5. 4
24	6. 4
25	3. 4
B. VI.	P. B.
Pr. 1	7. 4
2	12. 4
3	18. 4
4	14. 4
5	17. 4
6	15. 4
7	16. 4
8	19. 4
9	11. 5
10	15. 5
11	12. 5
12	13. 5
13	14. 5
14 }	Cor. to
15 }	25. 4
16 } 17 }	10. 4
18	11. 6
19	24. 4
20	26. 4
22	28. 4
23	25. 4
25	13. 6
27 } 28 }	17. 5
29	18. 5
30	19. 5
31	29. 4
B. XI.	P. B.
Pr. 3	1. 7
4	2. 7

Euclid.	Theſe El.
B. XI.	P. B.
Pr. 5	Cor. to 2. 7
6	4. 7
8	5. 7
9	8. 7
10	9. 7
12	3. 7
14	7. 7
15	10. 7
16	11. 7
18	6. 7
19	13. 7
24	16. 7
25	21. 7
28 } 29 } 30 } 31 }	17, 19 and 20 of the 7th, and Cor. 3. to 8. 8.
32	21. 7
33	9. 8
34	23. 7
35	14. 7
37	14. 7 { 25. 7
B. XII.	P. B.
Pr. 2	3. 8
5 } 6 }	Cor. 4 to 8. 8
7	7. 8
8	9. 8
9 } 10 } 11 } 12 } 13 } 14 } 15 }	Cor. 3, 4 and 6, to 8. 8
18	Cor. to 11. 8

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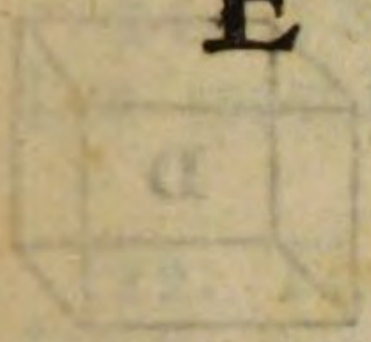


ELEMENTS

OF

GEOMETRY.

BOOK I.

DEFINITIONS.

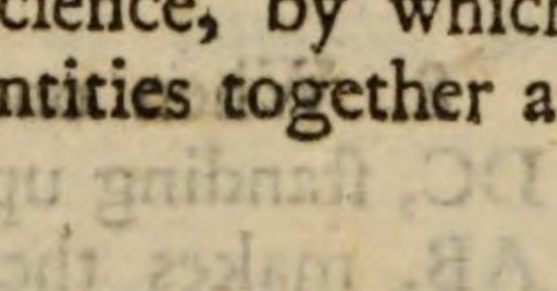
1. GEOMETRY is that ſcience, by which we compare ſuch quantities together as have extenſion.

Extenſion is diſtinguiſhed into length, breadth, and thickneſs.

2. A Line is that, which has length without breadth.

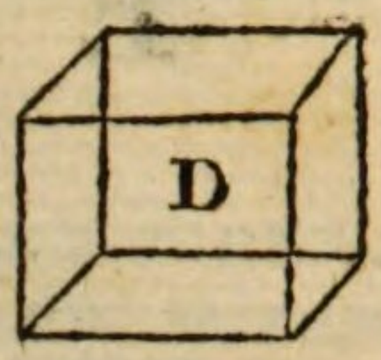
The terms, bounds, or extremes of a Line, are points.

3. A Surface is that, which has length and breadth, only, as C.



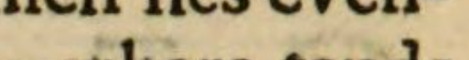
The bounds of a Surface are lines.

4. A Solid is that, which has length, breadth, and thickneſs, as D.



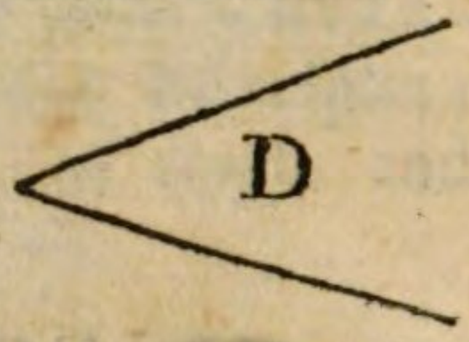
The bounds of a Solid are ſurfaces.

5. A Right (or ſtrait) line is that, which lies evenly between its extremes, or which every-where tends the ſame way, as AB.

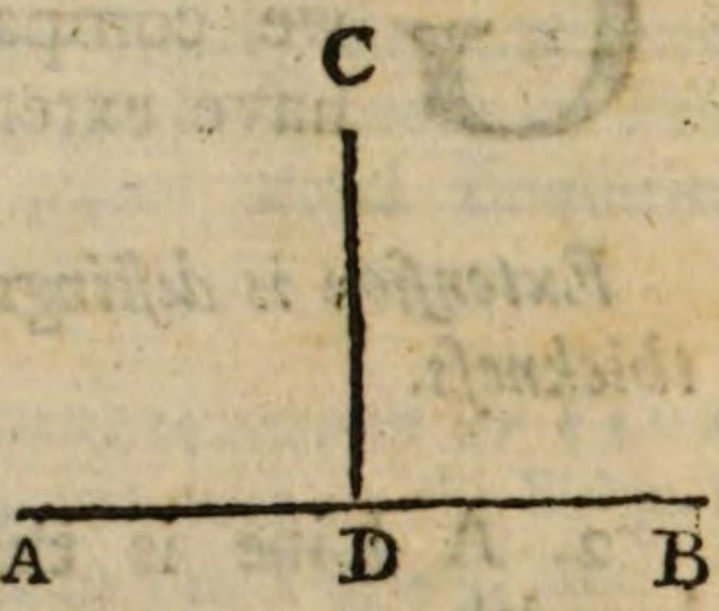


6. A Plane-ſurface is that, which is every-where perfectly flat and even, or which touches, in every part, any right-line extended between points anywhere taken in that ſurface.

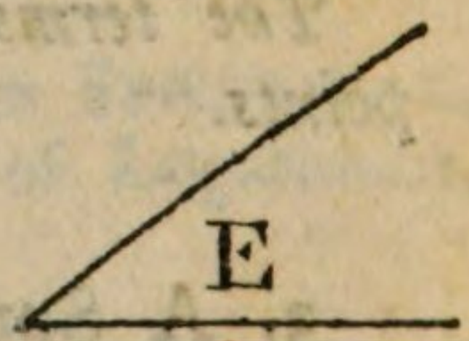
7. An Angle is the inclination, or opening of two right-lines meeting in a point, as D.



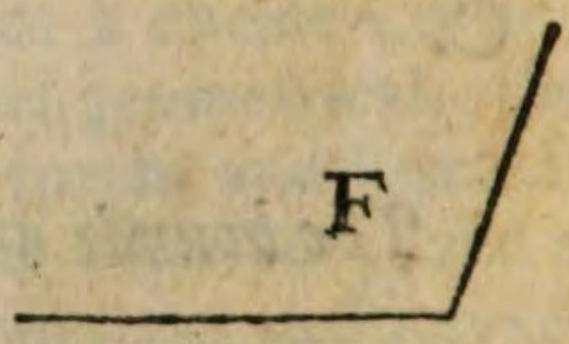
8. When one right-line DC, ſtanding upon another AB, makes the angles on both ſides equal, thoſe angles are called right-angles; and that line CD is ſaid to be perpendicular to the other AB on which it inſiſts.



9. An Acute-angle is that, which is leſs than a right-angle, as E.



10. An Obtuſe-angle is that, which is greater than a right-angle, as F.

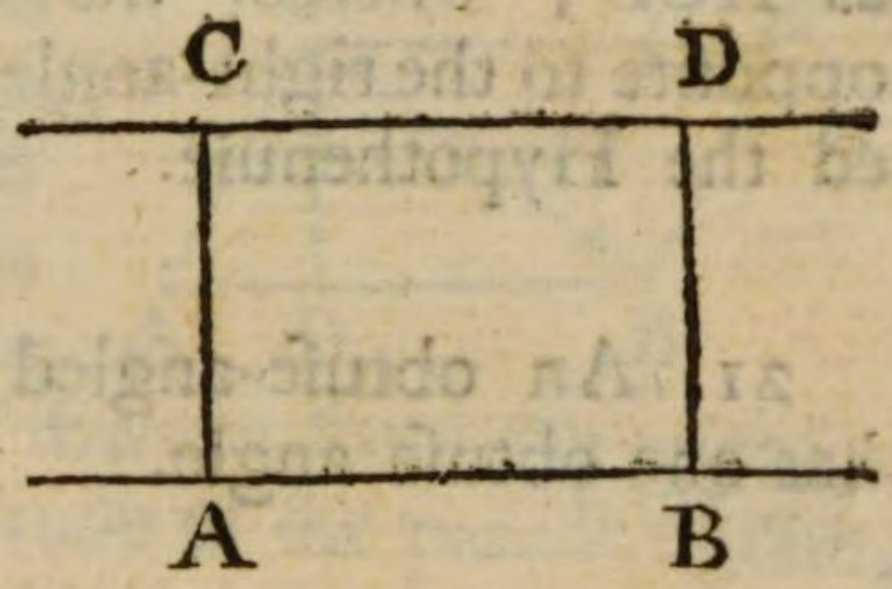


11. The

11. The distance of two points, is the Right-line reaching from the one to the other.

12. The distance of a point from a line, is a Right-line drawn from that point, perpendicular to, and terminating in, the line given.

13. Parallel (or equidistant) right-lines AB, CD are such, which being in the same plane-surface, if infinitely produced, would never meet.

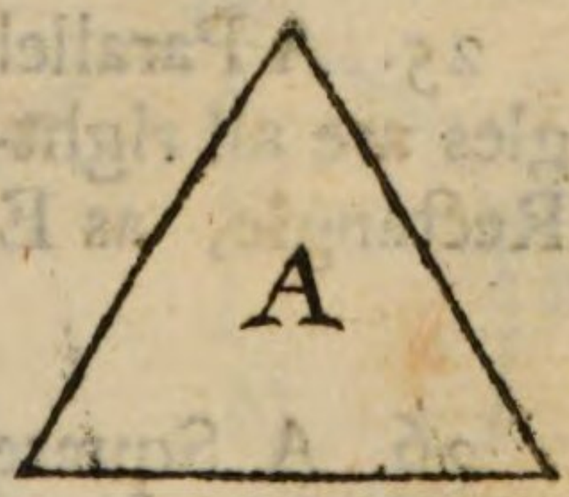


14. A Figure is a bounded space, and is either a surface, or a solid.

15. A right-lined plane Figure is that, formed in a plane-surface, whose terms, or bounds, are right-lines.

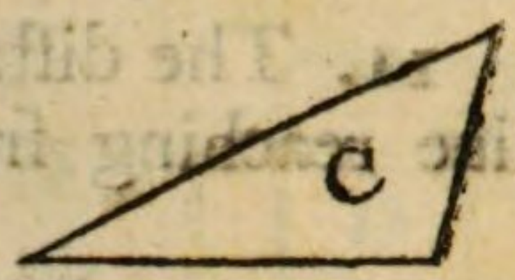
16. All plane Figures bounded by three right-lines, are called Triangles.

17. An equilateral Triangle is that whose bounds or sides are all equal, as A.

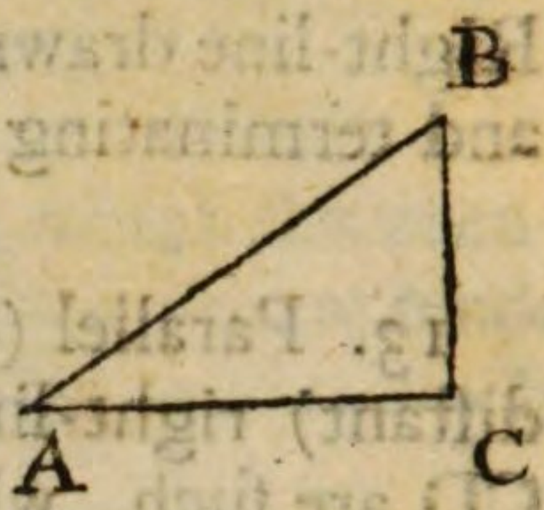


18. An isosceles Triangle is, when two sides are equal, as B.

19. A ſcalene Triangle is, when all the three ſides are unequal, as C.



20. A right-angled Triangle is that, which has one right-angle, as ACB; whereof the ſide AB oppoſite to the right-angle, is called the Hypothenuſe.

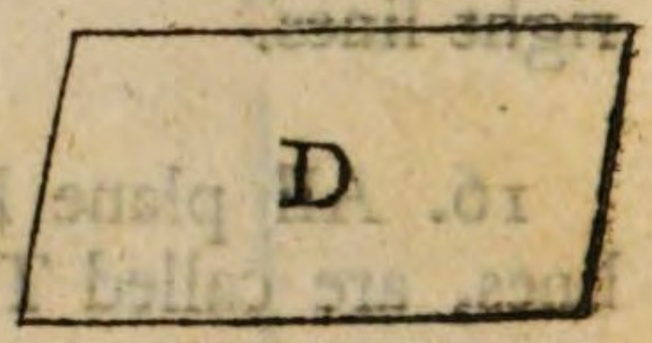


21. An obtuſe-angled Triangle is that, which has one obtuſe angle.

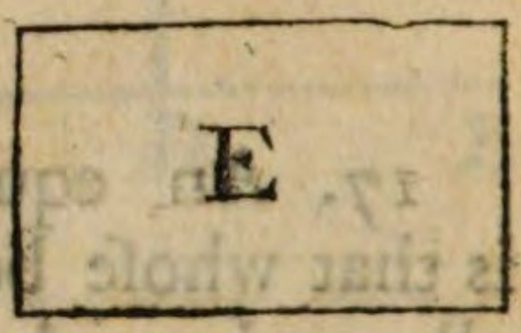
22. An acute-angled Triangle is that, which has all its angles acute.

23. Every plane Figure bounded by four right-lines, is called a Quadrangle, or Quadrilateral.

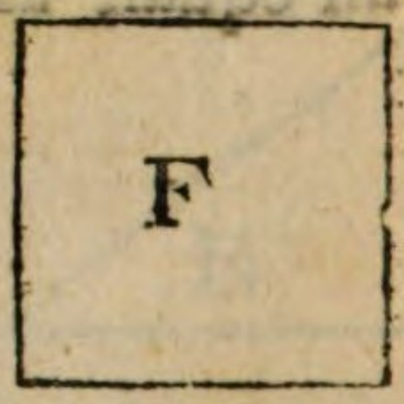
24. Any Quadrangle, whoſe oppoſite ſides are parallel, is called a Parallelogram, as D.



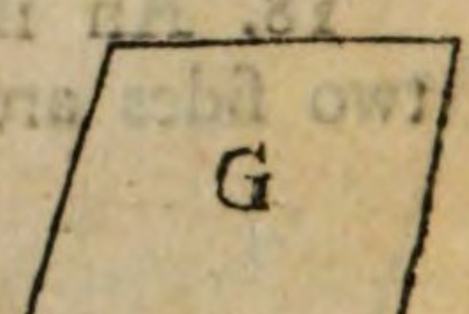
25. A Parallelogram, whoſe angles are all right-ones, is called a Rectangle, as E.



26. A Square is a parallelogram whoſe ſides are all equal, and its angles all right-ones, as F.



27. A Rhombus is a parallelogram whoſe ſides are all equal, but its angles not right, as G.

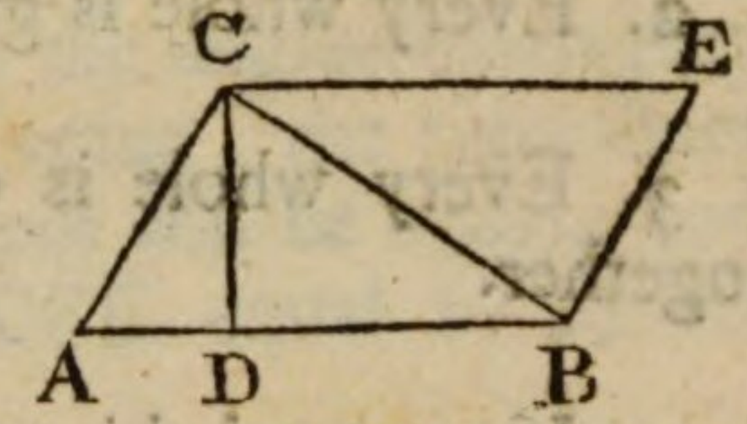


28. All

28. All other four-ſided figures, beſides theſe, are called Trapeziums.

29. A right-line joining any two oppoſite angles of a four-ſided figure, is called a Diagonal.

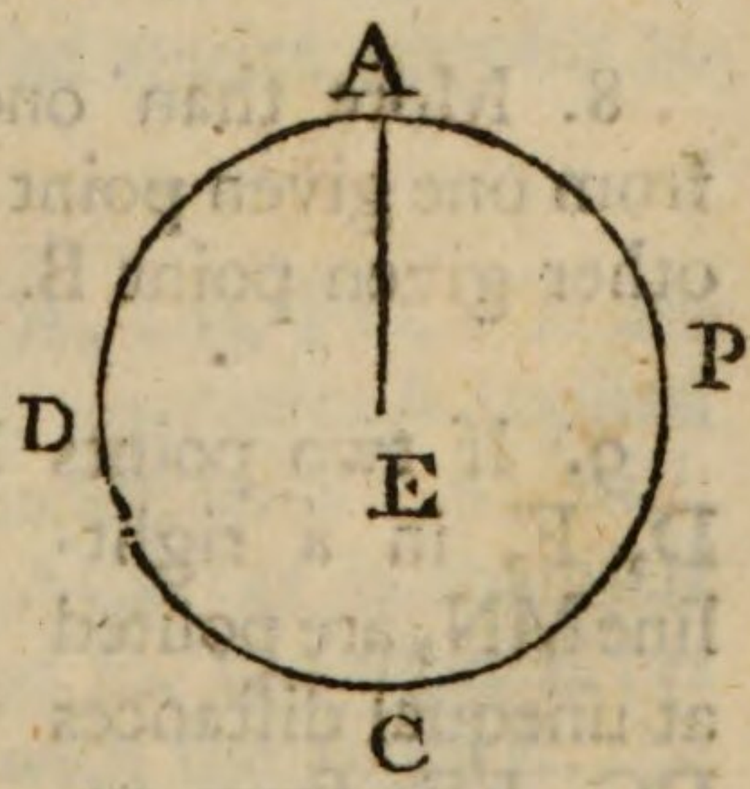
30. That ſide AB upon which any parallelogram ACEB, or triangle ACB is ſuppoſed to ſtand, is called the baſe; and the perpendicular CD falling thereon from the oppoſite angle C, is called the altitude of the parallelogram, or triangle.



31. All plane figures contained under more than four ſides, are called Polygons; whereof thoſe having five ſides, are called Pentagons; thoſe having ſix ſides, Hexagons; and ſo on.

32. A Regular Polygon is one whoſe angles, as well as ſides, are all equal.

33. A Circle is a plane figure, bounded by one curve-line APCD, called its circumference, everywhere equally diſtant from a point E within the circle, called the center thereof.



34. The Radius of a circle, is the diſtance of the center from the circumference, or a right-line EA drawn from the center, to the circumference.

AXIOMS, *or* Self-evident Truths.

1. Things, equal to one and the ſame thing, are alſo equal to each other.

2. Every whole is greater than its part.

3. Every whole is equal to all its parts taken together.

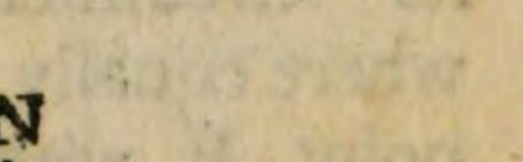
4. If to equal things, equal things be added, the wholes will be equal.

5. If from equal things, equal things be taken away, the remainders will be equal.

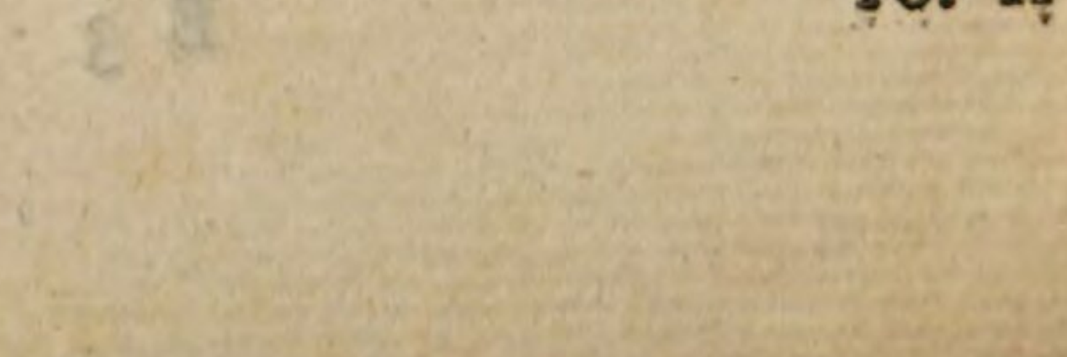
6. If to, or from unequal things, equal things be added, or taken away, the ſums, or remainders, will have the ſame difference, as the unequal things firſt propoſed,

7. All right-angles are equal to one another.

8. More than one right-line cannot be drawn from one given point A to another given point B.

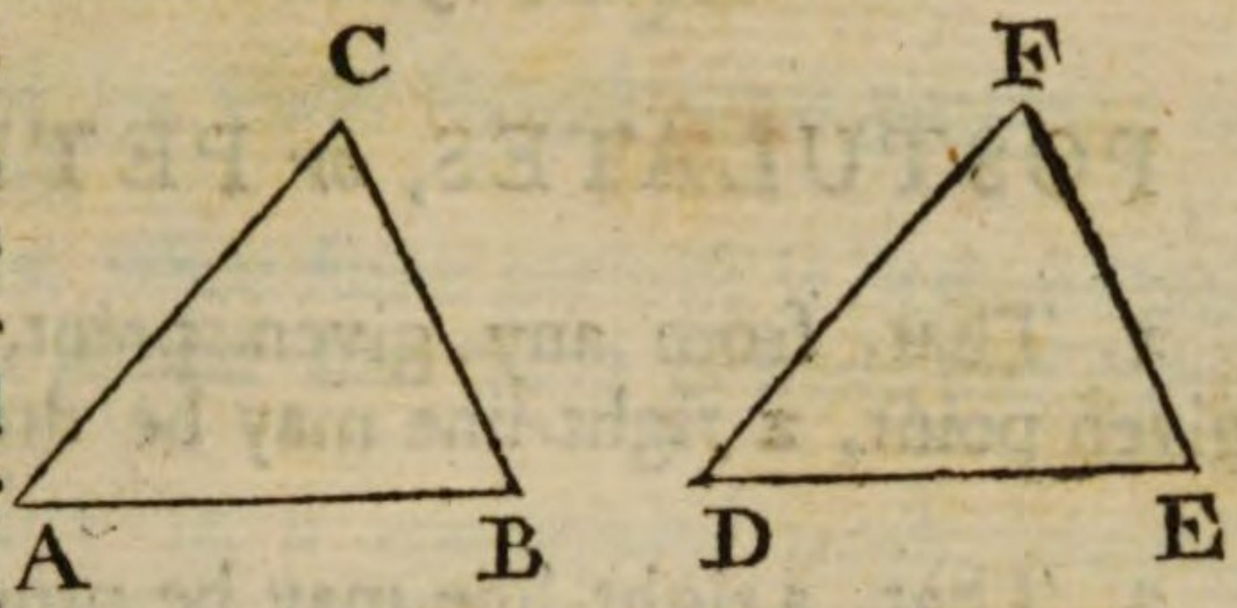


9. If two points D, F, in a right-line MN, are poſited at unequal diſtances DC, FE, from another right-line AB in the ſame plane-ſurface; thoſe two lines, being infinitely produced, on the ſide of the leaſt diſtance EF, will meet each other.



10. If

10. If two right-lines CA, CB, making an angle C, be reſpectively equal to two other right-lines FD, FE, making an angle F, and the angles which they make C, and F be likewiſe equal; the right-lines AB, DE joining their extremes will be equal, and the two triangles ACB, DFE equal in all reſpects.



If this ſhould not appear ſufficiently evident for an axiom; conceive the triangle DFE to be removed, and ſo applied to the triangle ABC, that the point F may coincide with C, and the ſide FD fall upon the ſide CA; then, becauſe FD is ſuppoſed equal to CA, the point D will alſo fall upon A. And, the angle F being equal to the angle C, the ſide FE will fall upon CB; and conſequently the point E upon the point B, becauſe FE is ſuppoſed equal to CB. Therefore, ſeeing all the bounds of the two triangles coincide, it is manifeſt, that not only the baſes AB, DE, but the angles oppoſite to the equal ſides, are alſo equal.

When all the four lines CA, CB, FD, FE are equal; the triangle DFE, being *contrariwiſe* applied to ACD ſo that FE may coincide with CA, will, *alſo*, agree with the triangle ACB (as is manifeſt from the reaſoning above): and ſo, the angle E (as D did before) now coinciding with the angle A, the two angles E and D muſt neceſſarily be equal to each other, in this caſe, where the triangle DFE is an iſoſceles one.

POSTULATES, *or* PETITIONS.

1. That, from any given point, to any other given point, a right-line may be drawn.

2. That, a right-line may be produced, or continued out, at pleasure.

3. That, from any point as a center, with a radius equal to any right-line assigned, a circle may be described.

4. That, a right-line may be drawn perpendicular to another, at any point assigned; and that it is also possible for to make a right-line, or a right-lined angle, equal to any right-line, or right-lined angle assigned, or to the half thereof.

This fourth Postulate *is added, more for the sake of making the proper references, than through absolute necessity: since, what is here barely assumed as possible, is effected, and actually demonstrated, in the beginning of the Fifth Book, intirely independent of every thing but* Axioms *and the other Postulates, above laid down. It may also be proper to note here, that, though these* Postulates *are not always quoted, it will be easy to perceive where, and in what sense, they are to be understood.*

NOTES *and* OBSERVATIONS, *with the significations of Signs used in this Tract.*

A PROPOSITION is, when something is, either, proposed to be done, or to be demonstrated, and is either a Problem, or a Theorem.

A PROBLEM is, when something is proposed to be done.

A THEO-

A THEOREM is, when ſomething is propoſed to be demonſtrated.

A LEMMA is, when ſome premiſe is demonſtrated, in order to render the thing in hand the more eaſy.

A COROLLARY is, a conſequent truth, gained from ſome preceding truth, or demonſtration.

A SCHOLIUM is, when remarks and obſervations are made upon ſomething going before.

The ſignification of SIGNS.

The ſign $=$, denotes that the quantities betwixt which it ſtands, are equal.

The ſign $\sqsubset$, denotes that the quantity preceding it, is greater than that which comes after it.

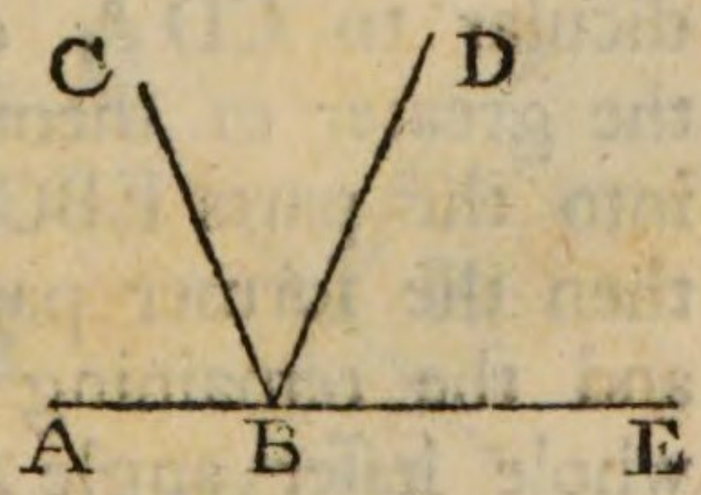
The ſign $\sqsupset$, denotes that the quantity preceding it, is leſs than that which comes after it.

The ſign $+$, denotes that the quantity which it precedes, is to be added.

The ſign $-$, denotes that the quantity which it precedes, is to be taken away or ſubtracted.

A figure, or number, prefixed to any quantity, ſhews how often that quantity is to be taken, or repeated; as 5A ſhews, that the quantity repreſented by A, is to be taken 5 times.

When ſeveral angles are formed about the ſame point (as at B), each particular angle is deſcribed by three letters, whereof the middle one ſhews the angular point, and the other two, the lines that form the angle: thus CBD or DBC ſignifies the angle formed by the lines CB and DB.



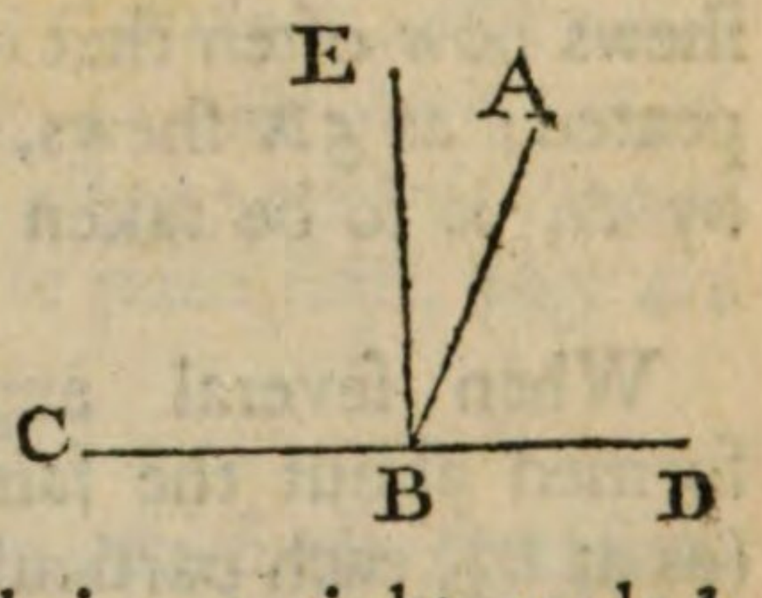
When,

When, in any demonſtration, you meet with ſeveral quantities joined the one to the other continually by the mark of equality (=), the concluſion drawn from thence, is always gathered from the firſt and laſt of them; which are equal to each other, by virtue of the firſt axiom. Thus if A = B = C = D, then will the firſt (A) and the laſt (D) be equal to each other.

Alſo, when in the quotations you meet with two numbers, the firſt ſhews the propoſition, and the ſecond the book. Moreover, Ax. denotes axiom; Poſt. poſtulatum; Def. definition; Hyp. hypotheſis. Note alſo, that, when-ever the word *Line* occurs, without the addition of either *right*, or *curved*, a right-line is always underſtood: and that, when a line is ſaid to be drawn to, or from an angle, the angular point is meant.

THEOREM I.

A line (AB) *ſtanding upon another line* (CD) *makes with it two angles* (ABC, ABD) *which, taken together, are equal to two right-angles.*



If the angles ABC, ABD are equal, it is plain they
[a] Def. 8. make two right-angles [a]; if unequal, let BE be perpen-
[b] Poſt. 4. dicular to CD [b], dividing the greater of them (ABC) into the parts EBC, EBA; then the former part EBC being a right-angle [a], and the remaining part EBA together with the whole leſſer angle ABD, equal to another right-
[c] Ax. 3. angle EBD [c]; the whole, of both the propoſed angles, taken together, muſt neceſſarily be equal
[d] Ax. 4. to two right-angles [d].

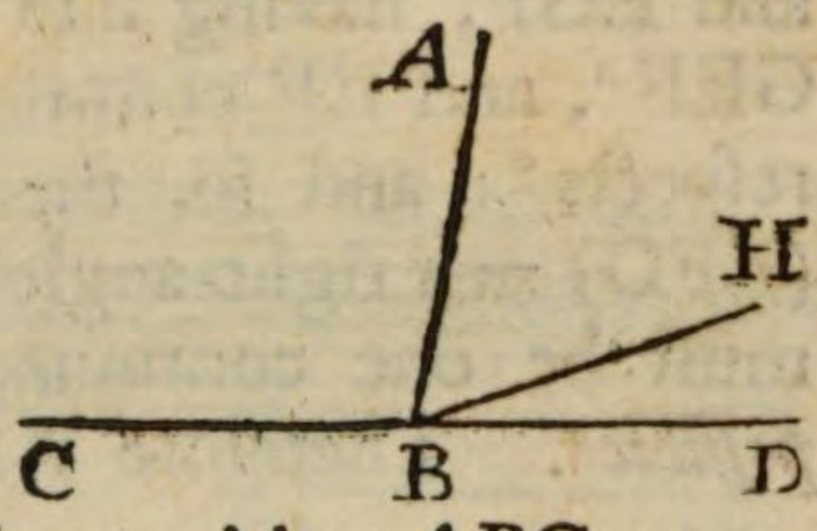
COROL-

COROLLARY.

Hence all the angles at the same point (B) on the same side of a right-line (CD) are equal to two right-angles [c]. [c] Ax. 3.

THEOREM II.

If one line (AB) *meeting two others* (BC, BD) *in the same point* (B), *makes two angles with them* (ABC, ABD) *which together are equal to two right-angles; these lines* (BC, BD) *will form one continued right-line.*

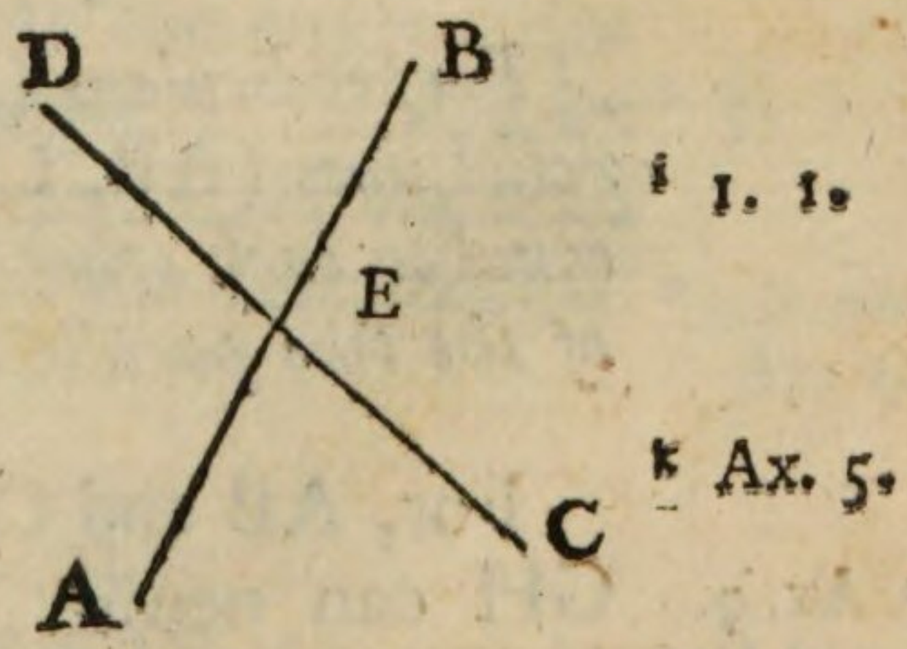


For, *if possible*, let BH, and not BD, be the continuation of the right-line CB: then the angles ABC and ABH being = two right-angles [e] = ABC and ABD [f]; if from these equal quantities, ABC, common to both, be taken away, there will remain ABH = ABD [g]; *which is impossible* [h].

[e] 1. 1.
[f] Hyp.
[g] Ax. 5.
[h] Ax. 2.

THEOREM III.

The opposite angles (DEB, AEC), *made by two lines* (DC, BA) *intersecting each other, are equal.*



For DEB + DEA = two right-angles [i] = AEC + DEA; whence, by taking away DEA, common, there remains DEB = AEC [k].

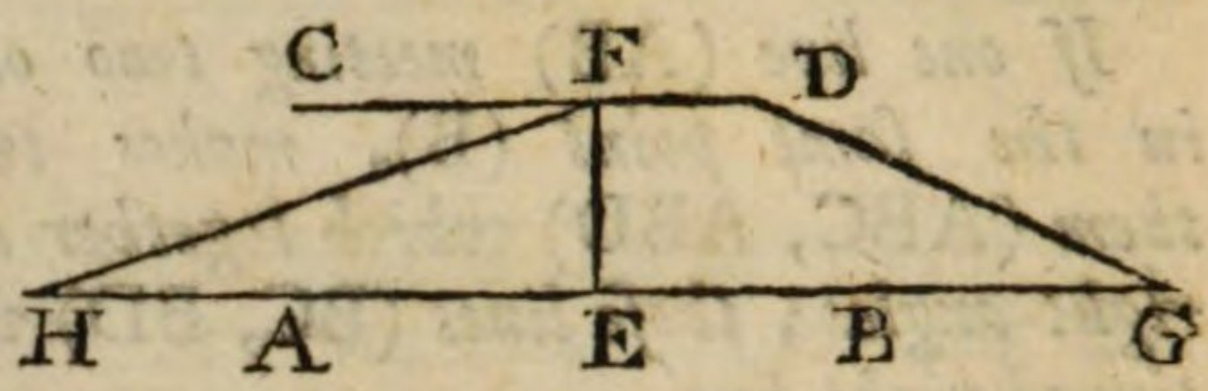
[i] 1. 1.
[k] Ax. 5.

THEO-

THEOREM IV.

Two right-lines (AB, CD) *perpendicular to one and the ſame right-line* (EF), *are parallel to each other.*

If you ſay, they are not parallel; then let them, when produced out, meet in ſome point, as G.



In EA, produced (if neceſſary) let there be taken EH = EG [l], and let the right-line FH be drawn [m]. The triangles EHF and EGF, having EH = EG, the angle HEF = GEF [n], and EF common, are therefore equal in all reſpects [o]: and ſo, the angle EFH being = EFG (EFD) = a right-angle [p], HFDG (as well as HEG) muſt be one continued right-line [q]: *which is impoſſible* [r]. Therefore AB and CD are parallels.

[l] Poſt. 4.
[m] Poſt. 1.
[n] Def. 8.
[o] Ax. 10.
[p] Hyp.
[q] 2. 1.
[r] Ax. 8.

SCHOLIUM.

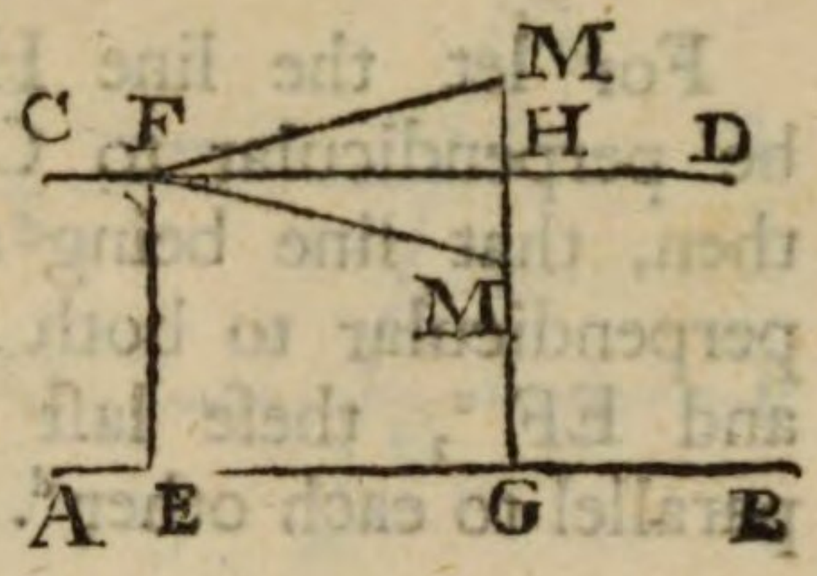
In this theorem, the *poſſibility* of parallel lines (or ſuch, which being infinitely produced, in the ſame plane, can never meet) is demonſtrated: for EF may be drawn perpendicular to AB [t]; and CFD, again, perpendicular to EF [t]; which laſt, it is demonſtrated, will be parallel to AB.

THEOREM V.

Perpendiculars (EF, GH) *to one* (AB) *of two parallel lines* (AB, CD) *terminated by thoſe lines, are equal to each other; and alſo perpendicular to the other of the two parallels* (CD).

For, AB and CD being parallel to each other, GH can neither be greater, nor leſs than EF [s]; and therefore muſt be equal to EF. If you ſay, that EF is not perpendicular to CD; then let FM be perpendicular to EF [t], meeting GH produced (if ne-

[s] Ax. 9. and Def. 13.
[t] Poſt. 4.



neceſſary) in M: ſo ſhall FM be parallel to AB[u]; and conſequently GM = EF[w] = GH; *which is impoſſible*[x]. Therefore EF is perpendicular to CD. And, by the ſame argument, GH is perpendicular to CD.

[u] 4. 1.

[w] 5. 1.

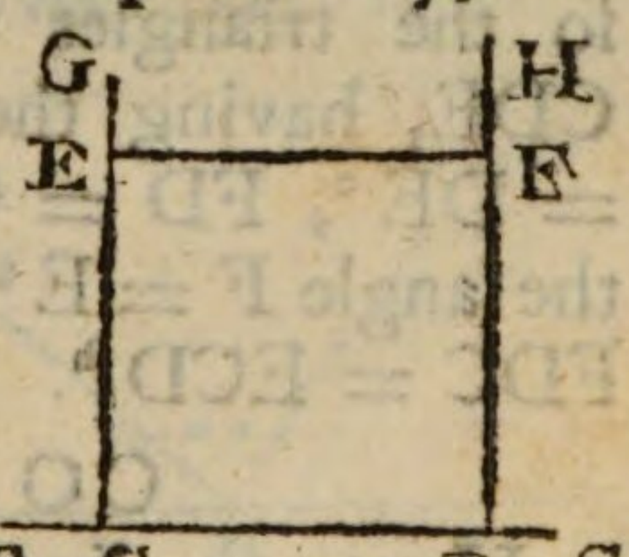
[x] Ax. 2.

COROLLARY.

Hence, through the ſame point F, more than one parallel cannot be drawn to the ſame line given AB.

SCHOLIUM.

From the preceding propoſition, the conſiſtence of the twenty-fifth definition, or the poſſibility, that all the properties aſcribed to a rectangle, can ſubſiſt together in the ſame figure, will appear, together with the method of conſtruction. For, at any two points C, D in a right-line RS, two perpendiculars CG, DH may be erected[y]; and a perpendicular to one of theſe, at any point E, meeting the other in F, may be drawn. The figure CEFD thus conſtructed will be a rectangle: for CE and DF are parallel[z]; as are alſo CD and EF[z]: therefore the angle F (as well as C, D, and E) is a right-angle[a]. If CE be made = CD, then will the rectangle CEDF have all its ſides equal[b]. Which anſwers to the definition of a ſquare.



[y] Poſt. 4.

[z] 4. 1.

[a] 5. 1.

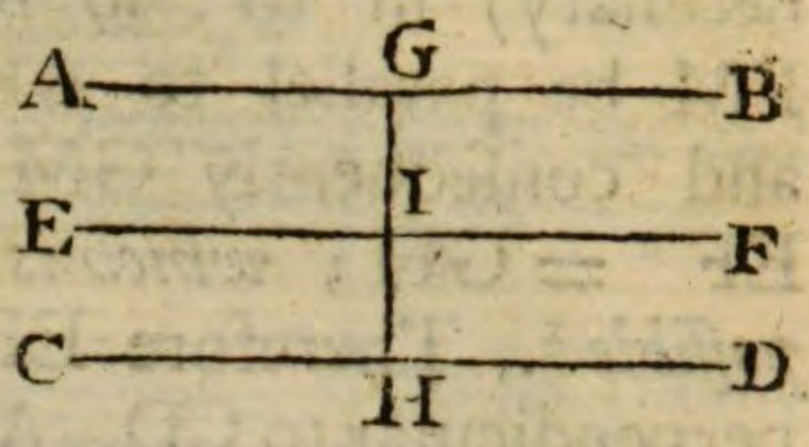
[b] 5. 1. and Ax. 1.

THEOREM VI.

Right-lines (AB, EF) *parallel to the ſame right-line* (CD) *are parallel to each other.*

For

For let the line HIG be perpendicular to CD: then, that line being alſo perpendicular to both AB and EF [c], theſe laſt are parallel to each other [d].

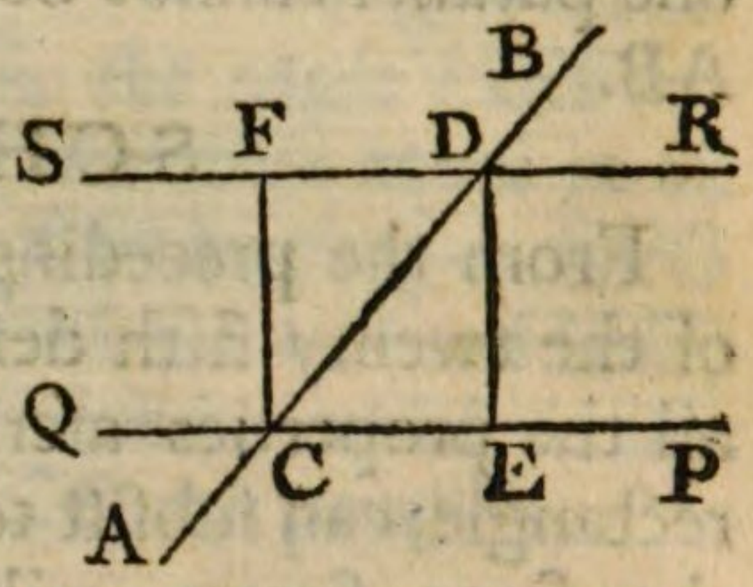


[c] 5. 1.
[d] 4. 1.

THEOREM VII.

A line (AB) *interſecting two parallel lines* (SR, QP) *makes the alternate angles* (SDC, PCD) *equal to each other.*

Let CF and DE be perpendicular to QP, and SR [e]; then theſe lines FC and DE are likewiſe parallels [f]; and ſo the triangles CFD and CDE, having the ſide CF = DE [e], FD = CE [e], and the angle F = E [g], they will alſo have the angle FDC = ECD [h].



[e] 5. 1.
[f] 4. 1.
[g] Ax. 7.
[h] Ax. 10.

COROLLARY I.

Hence, a line interſecting two parallel lines, makes the angles (BDR, BCP) on the ſame ſide, equal to each other: for BDR (= CDS [i]) = BCP [k].

[i] 3. 1.
[k] 7. 1.

COROLLARY II.

Hence, alſo, a line falling upon two parallel lines, makes the ſum of the two internal angles (SDC + QCD) on the ſame ſide of it, equal to two right-angles: for the angle SDC being = PCD, and PCD + QCD = two right-angles [l]; thence is SDC + QCD = alſo to two right-angles [m].

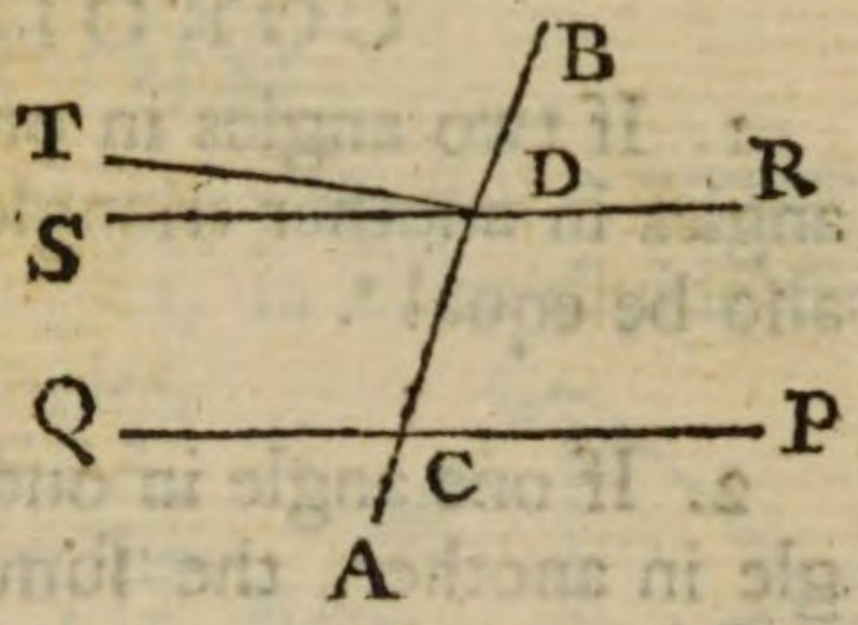
[l] 1. 1.
[m] Ax. 4.

THEOREM VIII.

If a line (AB) *interſecting two other lines* (PQ, RS), *makes the alternate angles* (DCP, CDS) *equal to each other; then are thoſe two lines parallel.*

For,

For, if poſſible, let ſome other line DT, and not DS, be parallel to PQ [n]; then muſt CDT = DCP [o] = CDS [p]: *which is impoſſible* [q].



[n] Sch. to 4. 1.
[o] 7. 1.
[p] Hyp.
[q] Ax. 2.

COROLLARY.

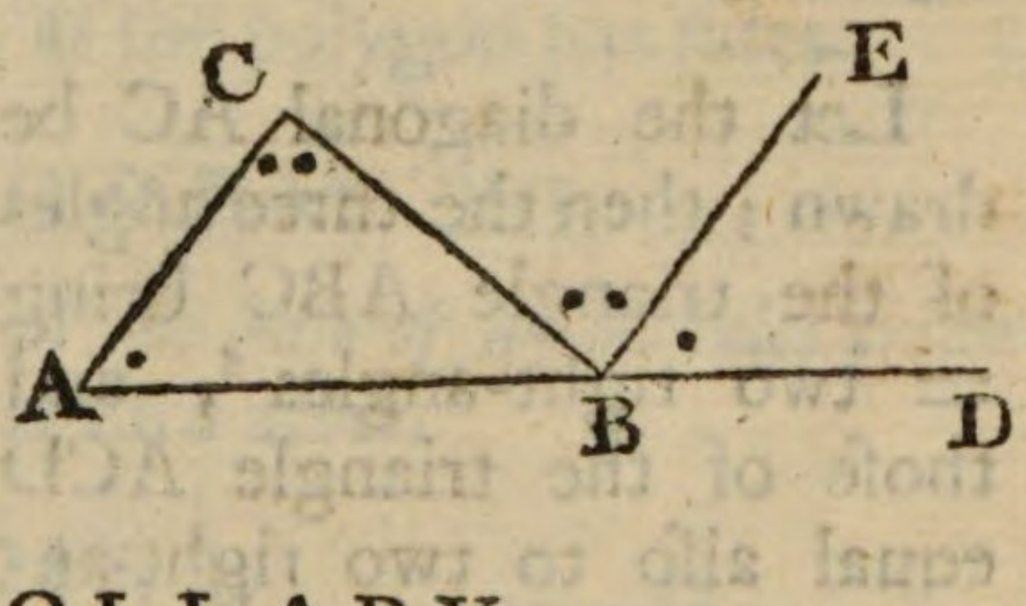
Hence, if a line falling on two others, makes the angles (BDR, BCP) above them, on the ſame ſide, equal to each other; then thoſe two lines are parallels: becauſe SDC = BDR [r].

[r] 3. 1.

THEOREM IX.

If one ſide (AB) *of a triangle* (ABC) *be produced, the external angle* (CBD) *will be equal to both the internal oppoſite angles* (A, C) *taken together.*

For, let BE be parallel to AC [s]; then will the angle C = CBE [t], and the angle A = DBE [u]; therefore C + A = CBE + DBE [x] = CBD [y].



[s] Sch. to 4. 1.
[t] 7. 1.
[u] Cor. to 7. 1.
[x] Ax. 4.
[y] Ax. 3.

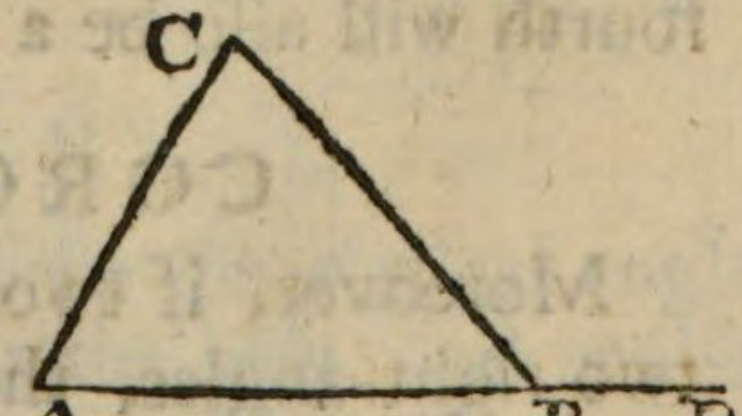
COROLLARY.

Hence the external angle of a triangle is greater than either of the internal, oppoſite angles.

THEOREM X.

The three angles of any plane triangle (ABC) *taken together, are equal to two right-angles.*

For, if AB be produced to D, then C + A = CBD [z], to which equal quantities let the angle CBA be added, then will C + A + CBA = CBD + CBA [a] = two right-angles [b].



[z] 9. 1.
[a] Ax. 4.
[b] 1. 1.

COROL-

COROLLARIES.

1. If two angles in one triangle, be equal to two angles in another triangle, the remaining angles will also be equal [c].

[c] Ax. 5.

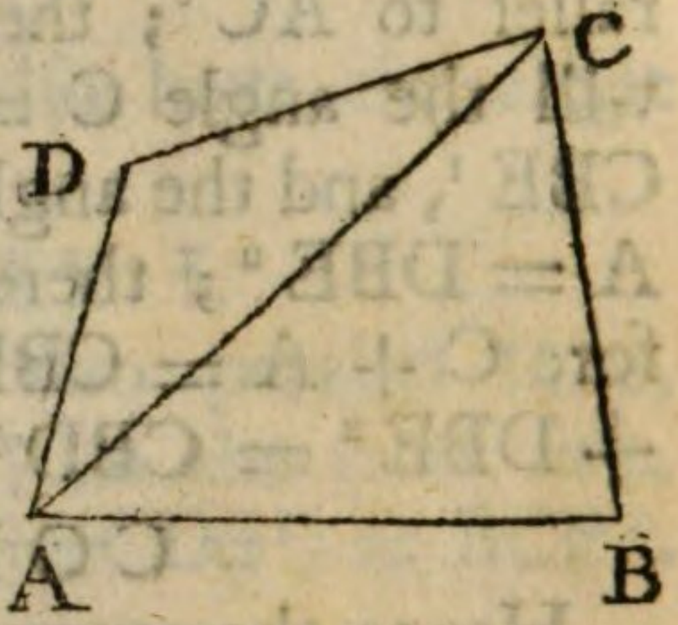
2. If one angle in one triangle, be equal one angle in another, the ſums of the remaining angles will be equal [c].

3. If one angle of a triangle be right, the other two taken together, will be equal to a right-angle.

4. The two leaſt angles, of every triangle, are acute.

THEOREM XI.

The four inward angles of a quadrangle (ABCD) *taken together, are equal to four right-angles.*



Let the diagonal AC be drawn; then the three angles of the triangle ABC being = two right-angles [d], and thoſe of the triangle ACD equal alſo to two right-angles [d]; it follows that the ſum of all the angles of both triangles, which make the four angles of the quadrangle, muſt be equal to four right-angles [e].

[d] 10. 1.

[e] Ax. 4.

COROLLARY I.

Hence, if three of the angles be right-ones, the fourth will alſo be a right-angle.

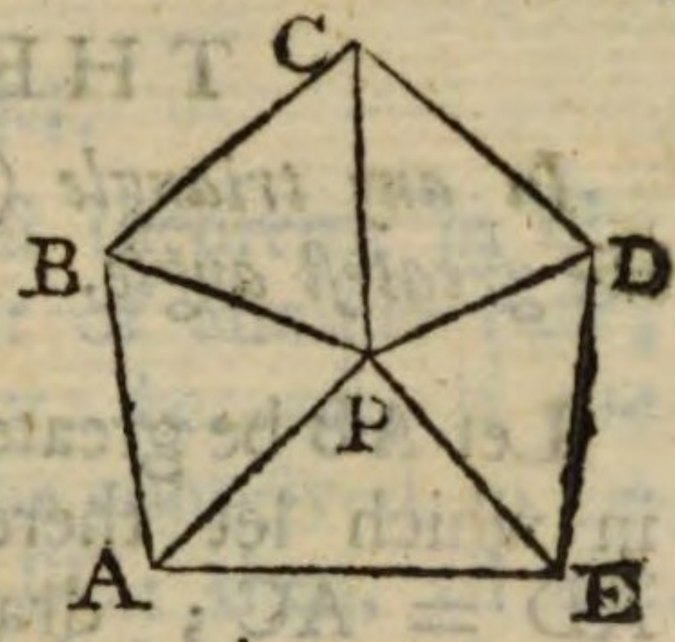
COROLLARY II.

Moreover, if two of the four angles, be equal to two right-angles, the remaining two together will likewiſe be equal to two right-angles.

SCHO-

SCHOLIUM.

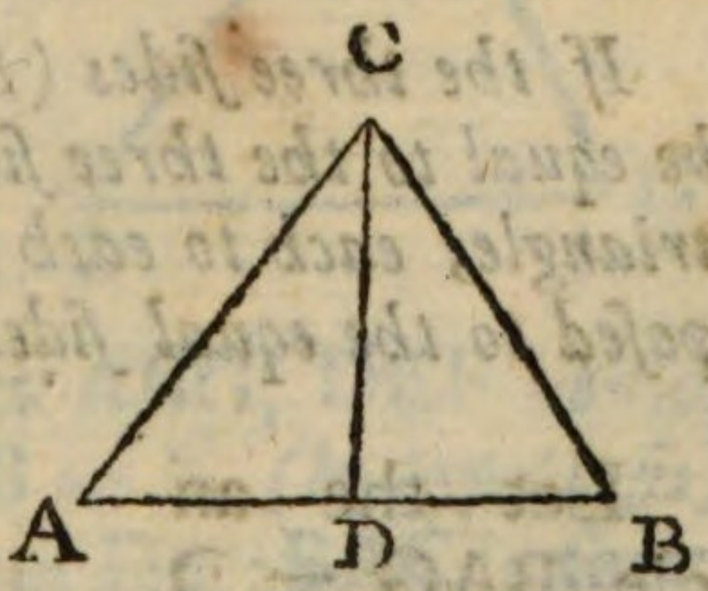
If from any point P, within a polygon ABCDE, lines be drawn to all the angles, ſo as to divide the whole into as many triangles, APB, BPC, CPD, DPE, EPA, as the polygon has ſides; the ſum of all the angles of theſe triangles (which together make up, or compoſe the angles of the polygon, over and above thoſe about the point P) will be equal to twice as many right-angles as the polygon has ſides (by 10. 1.) Therefore, ſeeing all the angles about the point P, whereby the angles of all the triangles exceed thoſe of the polygon, are equal to four right-angles, it is manifeſt, that all the angles of the polygon, taken together, will be equal to twice as many right-angles, wanting four, as the polygon has ſides.



THEOREM XII.

The angles (A, B) *at the baſe of an iſoſceles triangle* (ABC) *are equal to each other.*

For, let the line CD biſect, or divide the angle ACB into two equal parts ACD, BCD, and meet AB in D: then the triangles ACD, BCD, having AC = BC [f], CD common, and the angle ACD = BCD [g], will alſo have the angle A = B [h].



[f] Def. 18.
[g] Hyp.
[h] Ax. 10.

COROLLARY I.

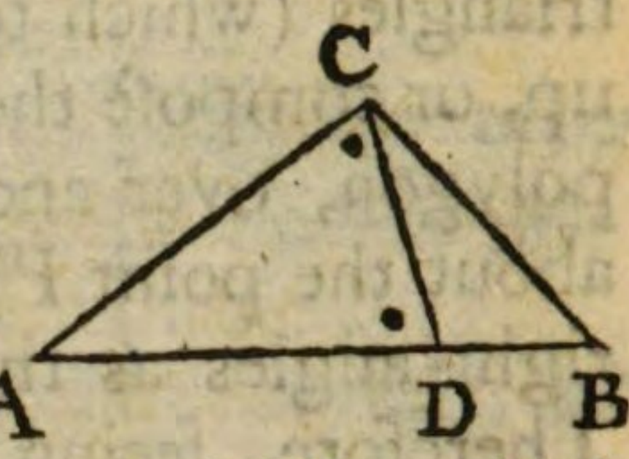
Hence, the line which biſects the vertical angle of an iſoſceles triangle, biſects the baſe, and is alſo perpendicular to it [h].

COROLLARY II.

Hence it appears alſo, that every equilateral triangle is likewiſe equiangular.

THEOREM XIII.

In any triangle (ABC) *the greateſt ſide ſubtends the greateſt angle.*



Let AB be greater than AC; in which let there be taken AD = AC; drawing CD. The triangle ADC being iſoſceles, the angles ACD and ADC are therefore equal [i]; whence ACB, which exceeds the former of them, muſt alſo exceed the latter ADC [k], and conſequently, much more exceed B, which is leſs than ADC [l].

[i] 12. 1.
[k] Ax. 2.
[l] Cor. to 9. 1.

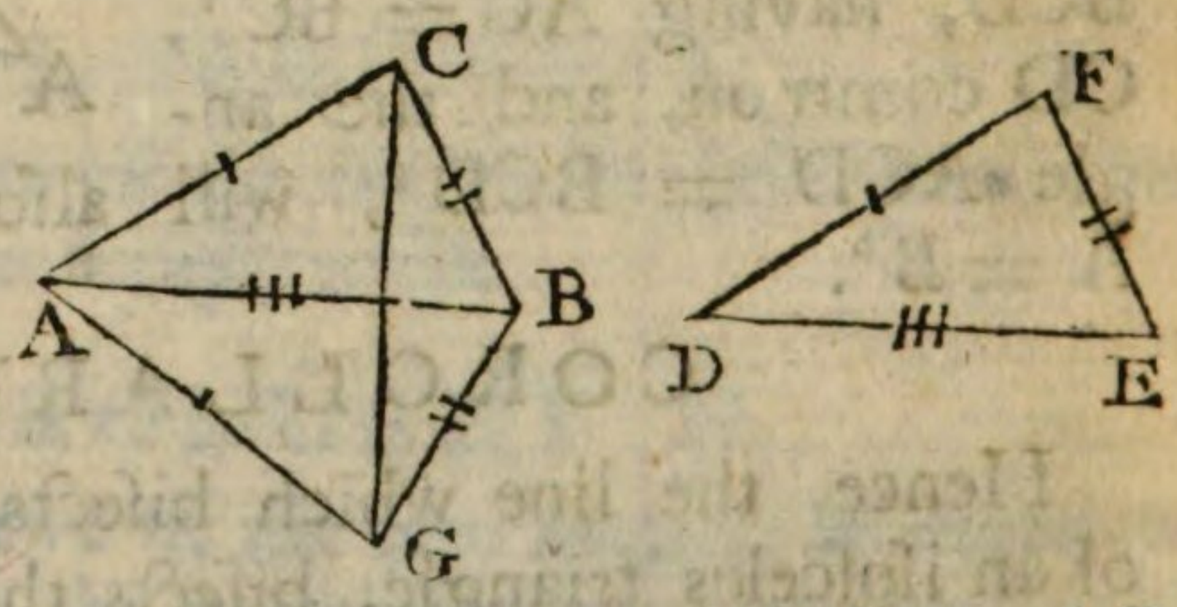
COROLLARY.

Hence, in any triangle, the ſide that ſubtends the greateſt angle, is the greateſt; becauſe ACB cannot be greater than B, unleſs AB is greater than AC [m].

[m] 13. 1.

THEOREM XIV.

If the three ſides (AB, AC, CB) *of one triangle, be equal to the three ſides* (DE, DF, FE) *of another triangle, each to each reſpectively; then the angles oppoſed to the equal ſides will alſo be equal.*



Let the angle BAG = D, AG = DF, and let GB and GC be drawn; ſo ſhall the triangles ABG and DEF be equal in all reſpects [m]: therefore, AG being = DF = AC [n], and BG = EF = BC [n], the

[m] Ax. 10.
[n] Hyp.

the angle ACG is also = AGC°, and BCG• = BGC°; and consequently ACB = AGB[p] = DFE: therefore the triangles ABC, DEF are equal in all respects[n]. • 12. 1. [p] Ax. 4 or 5.

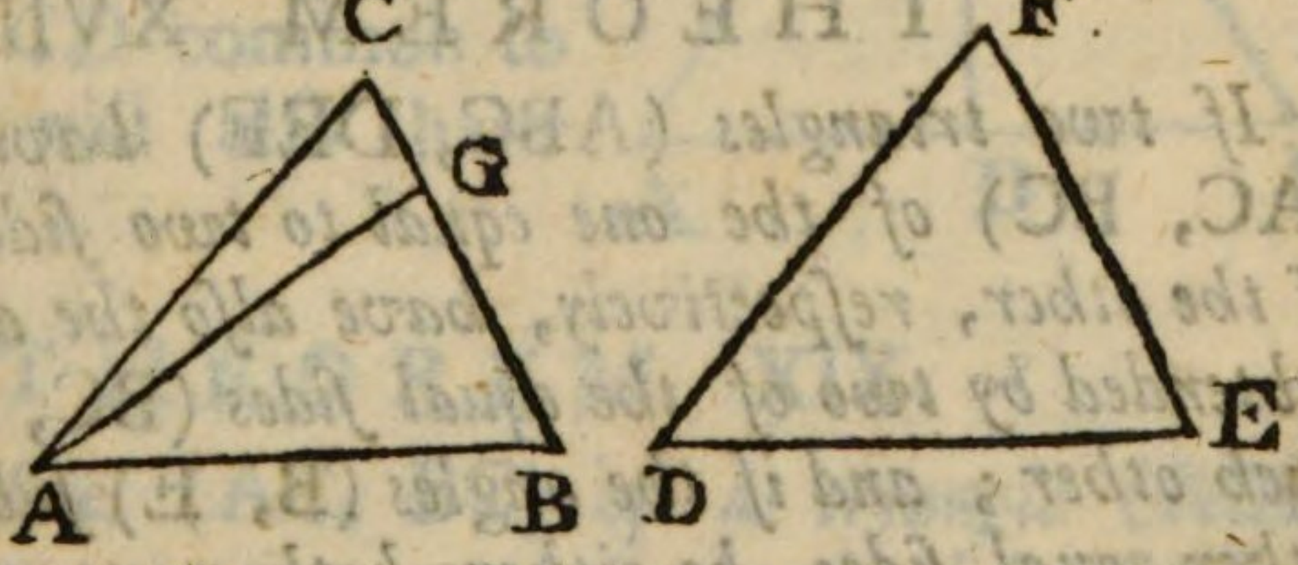
SCHOLIUM.

The demonstration of the last theorem, in obtuse-angled triangles, may admit of another case; which, however, is not necessary: because, if the triangle AGB (equal to DEF) be conceived to be formed on the longest side of ABC; then, all the angles CAB, CBA, GAB, GBA being acute[q], the line CG will, always, fall within the figure ACBG[r], as in the present case. [q] Cor. 4. to 10. 1. [r] Ax. 2.

THEOREM XV.

If two triangles (ABC, DEF) *mutually equiangular, have two corresponding sides* (AB, DE) *equal to each other, the other corresponding sides will also be equal.*

If you say BC is greater than EF; from BC let a part BG be taken = EF[s], and let AG be drawn. The triangles ABG, DEF having AB = DE, BG = EF, and B = E *(by hypothesis)*, will also have BAG = D[t]; but D = BAC[u]; therefore BAG = BAC[w]; *which is impossible.*



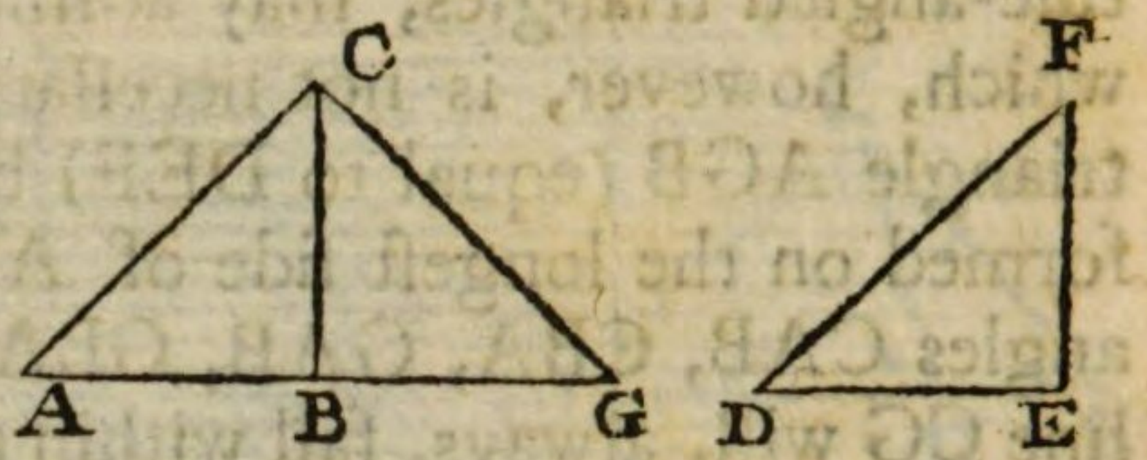
[s] Post. 4. [t] Ax. 10. [u] Hyp. [w] Ax. 1. and 2.

COROLLARY.

Hence, equiangular triangles, having any two corresponding sides equal, are equal to each other[x]. [x] Ax. 10. 1.

THEOREM XVI.

If two right-angled triangles (ABC, DEF) *having equal hypothenuſes* (AC, DF), *have two other ſides* (BC, EF) *likewiſe equal; the remaining ſides* (AB, DE) *will be equal, and the two triangles equal in all reſpects.*



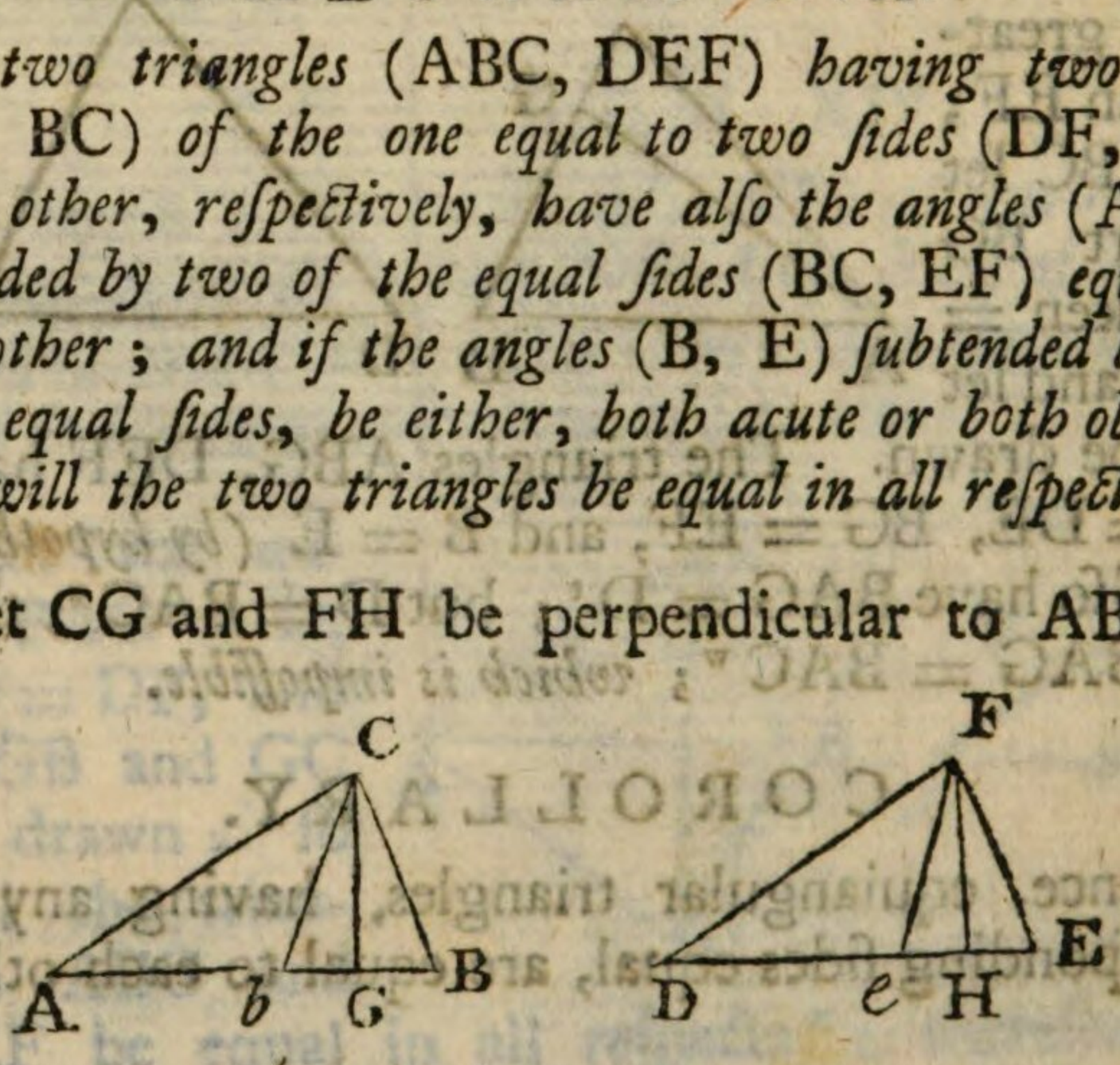
In AB produced, take BG = ED, and let GC be drawn: then, the triangles BCG and DEF, having BG = ED, BC = EF [y], and the angle CBG = E [x], will alſo have the angle G = D, and CG = DF [a] = AC [y]: whence, the triangle ACG being iſoſceles, the angle G, or D, will be = A [b]; and conſequently F alſo = ACB [c]: therefore the triangles ABC and DEF, being mutually equiangular, and having AC = DF, they are equal in all reſpects [d].

[y] Hyp.
[x] Ax. 7.
[a] Ax. 10.
[b] 12. 1.
[c] Cor. 1. to 10. 1.
[d] 15. 1.

THEOREM XVII.

If two triangles (ABC, DEF) *having two ſides* (AC, BC) *of the one equal to two ſides* (DF, EF) *of the other, reſpectively, have alſo the angles* (A, D) *ſubtended by two of the equal ſides* (BC, EF) *equal to each other; and if the angles* (B, E) *ſubtended by the other equal ſides, be either, both acute or both obtuſe; then will the two triangles be equal in all reſpects.*

Let CG and FH be perpendicular to AB and



DE; then, the angle AGC being = DHF [x],

A = D, and the ſide AC = DF [f], CG will alſo be = FH [g]; whence, CB being = FE [f], the angles GBC and HEF are likewiſe equal [h]: and ſo, the triangles ABC and DEF, being mutually equiangular [i], and having the ſides AC and DF equal, are equal in all reſpects [g].

[f] Hyp.
[g] 15. 1.
[h] 16. 1.
[i] Cor. 1. to 10.

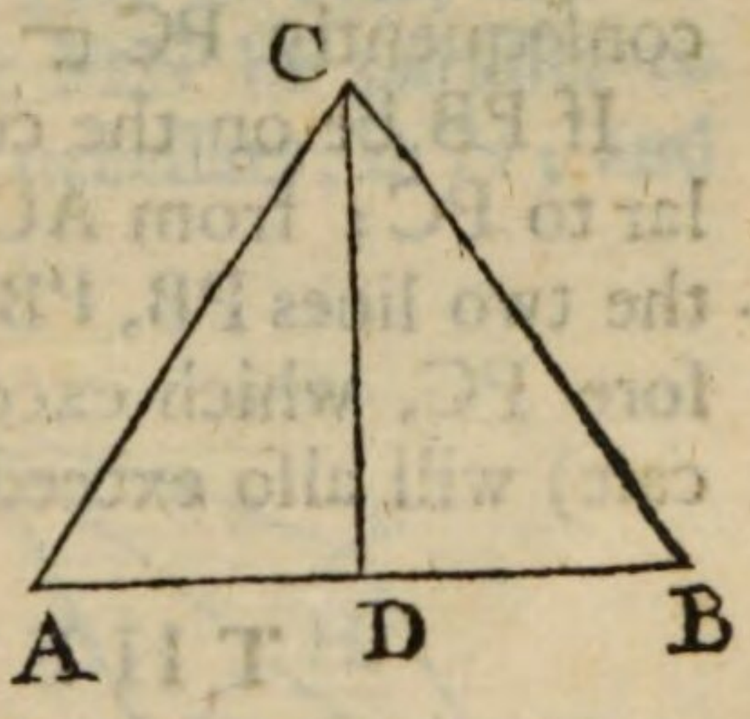
The demonſtration is the ſame, when both the angles are obtuſe, as in the triangles A*b*C, D*e*F: for, if C*b* (= CB = FE) = F*e*, the angles G*b*C and H*e*F being equal *(as before)*, the angles A*b*C and D*e*F will likewiſe be equal [k].

[k] 1. 1. and Ax. 5. 1.

THEOREM XVIII.

If two angles (A, B) *of a triangle* (ABC) *be equal, the ſides* (BC, AC) *ſubtending them will likewiſe be equal.*

Let CD biſect the angle ACB, and meet AB in D: then the triangles ACD, BCD being equiangular [k], and having CD common to both, they will alſo have AC = BC [l].

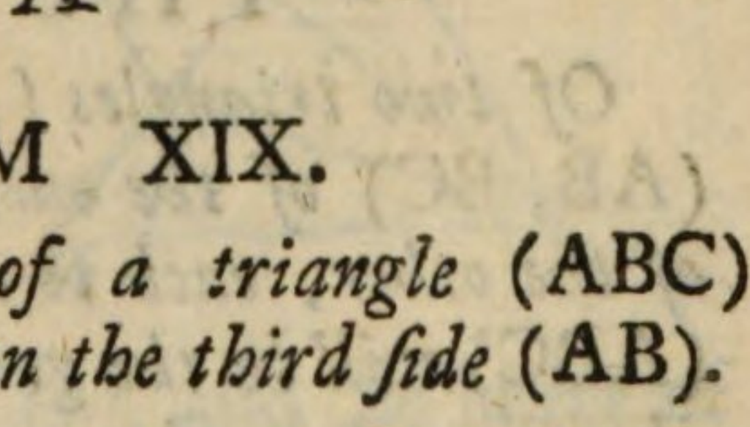


[k] Cor. 1. to 10.
[l] 15. 1.

THEOREM XIX.

Any two ſides (AC, BC) *of a triangle* (ABC) *taken together, are greater than the third ſide* (AB).

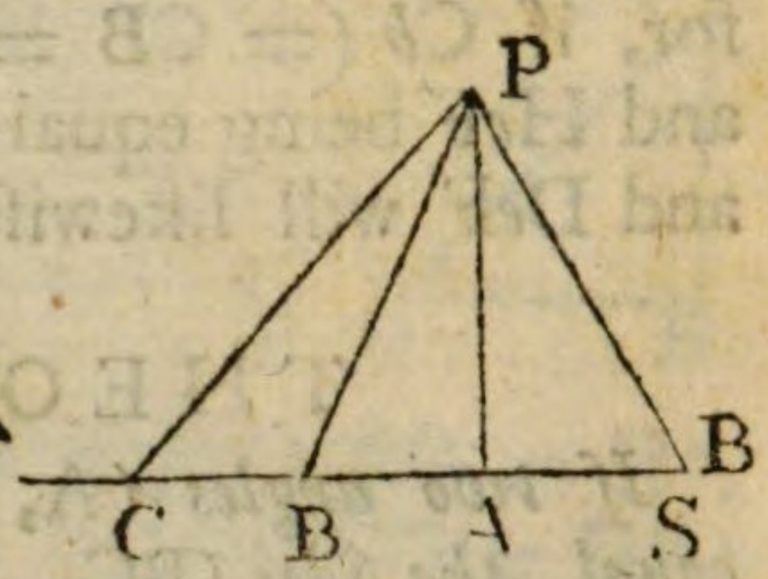
In BC produced, let there be taken CD = CA, and let AD be drawn. The angles D and DAC are equal [m]; therefore BAD, which exceeds the latter [n], muſt alſo exceed the former D; and conſequently BD (or BC + AC) muſt exceed AB [o].



[m] 12. 1.
[n] Ax. 2.
[o] Cor. to 13. 1.

THEOREM XX.

Of all the right-lines (PA, PB, PC) *falling from a given point* (P) *upon an infinite right-line* (RS), *that* (PA) *is the leaſt which is perpendicular to it; and, of the reſt, that* (PB) *which is the neareſt the perpendicular is leſs than any other* (PC) *at a greater diſtance.*



For BAP being a right-angle [p], ABP will be acute [q], and therefore AP ⊐ BP [r].

[p] Hyp.
[q] Cor. 4. to 10. 1.
[r] Cor. to 13. 1.

Again, when PB and PC are both on the ſame ſide of the perpendicular PA; then is CBP ⊏ right-angle [s] ⊏ BCP [q], and conſequently PC ⊏ PB.

[s] Cor. to 9. 1.

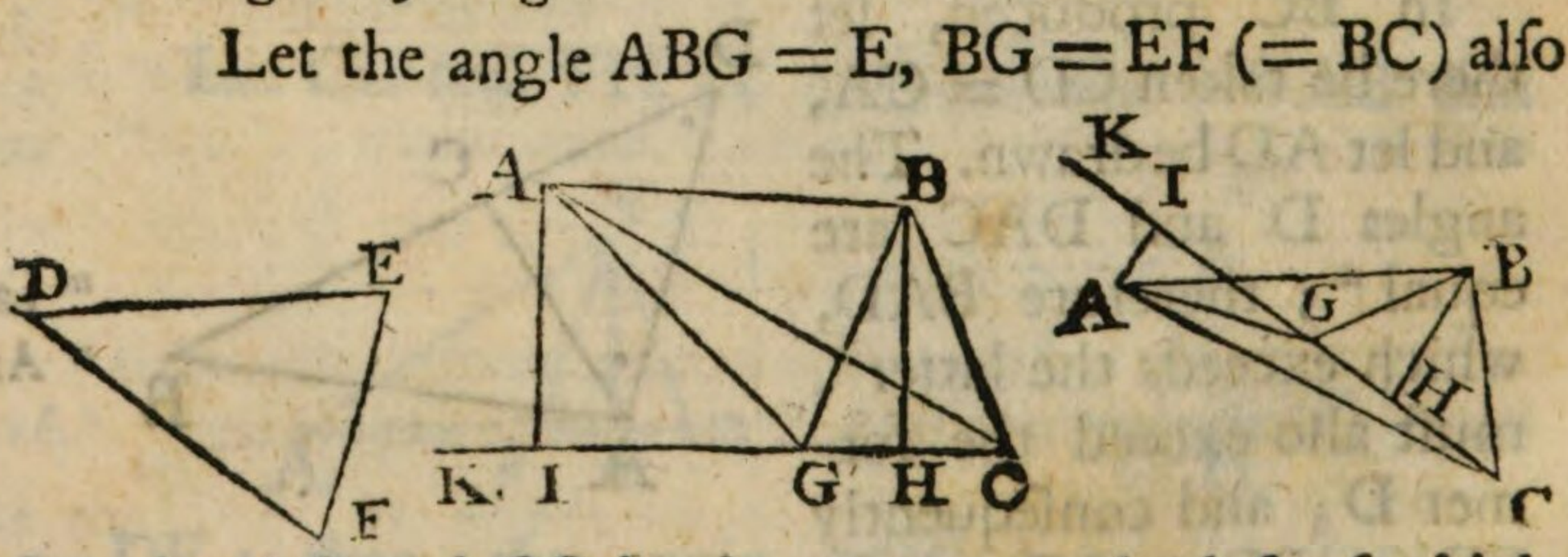
If PB be on the contrary ſide of the perpendicular to PC; from AC, let AB be taken = AB; then the two lines PB, PB will be alſo equal [t]; and therefore PC, which exceeds the one (by the preceding caſe) will alſo exceed the other.

[t] Ax. 10.

THEOREM XXI.

Of two triangles (ABC, DEF) *having two ſides* (AB, BC) *of the one, equal to two ſides* (DE, EF) *of the other, each to each reſpectively, the baſe of that* (ABC) *will be the greateſt, which is ſubtended under the greateſt angle.*

Let the angle ABG = E, BG = EF (= BC) alſo



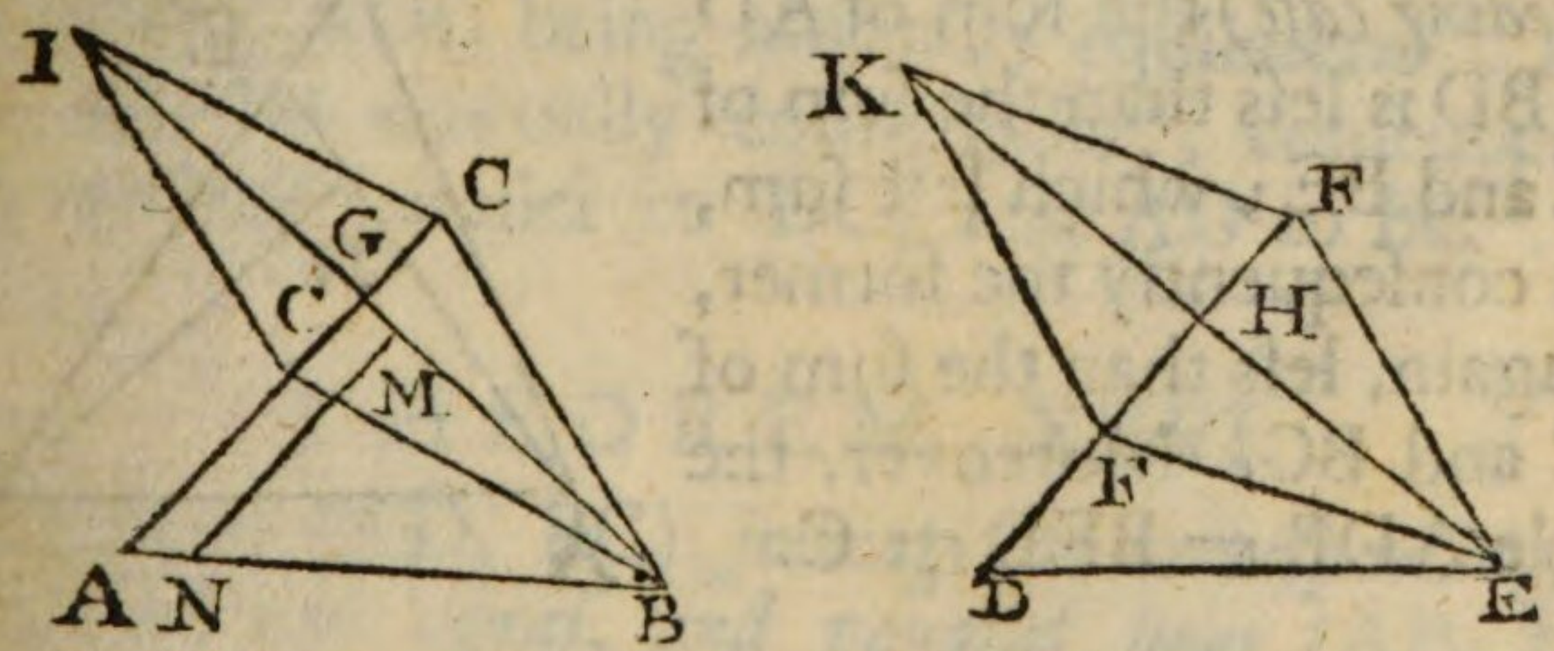
let AG and CG be drawn, upon the laſt of which, produced,

produced, let fall the perpendiculars BH and AI [u]. [u] 4. 1.
Since BG = BC [w], and, consequently, GH = HC [x], [w] Hyp.
it is evident, that GI (whether the point I be consi- [x] 16. 1.
dered as falling between G and K, or between G [z] Ax. 2.
and H) will be less than CI [z]; and therefore AG, [a] Ax. 10.
or its equal DF [a], also less than AC [b]. [b] 20. 1.

THEOREM XXII.

Of two triangles (ABC, DEF), *having one angle* (BAC) *in the one equal to one angle* (EDF) *in the other, and the sides* (BC, EF) *opposed to them also equal, that* (ABC) *will have the greatest base, whereof the opposite angle* (ACB) *differs the least from a right-angle.*

Let BG and EH be perpendicular to AC and DF, in which produced, take HK = HE, GI = GB, and BM = EH; also let MN be parallel to GA, meeting AB, produced if necessary, in N; and let CI and KF be drawn.

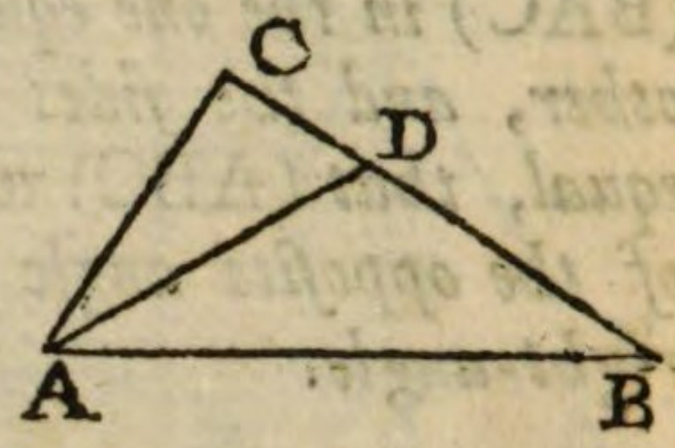


The angle ICG being = BCG [d], and the latter of [c] Hyp.
these greater than EFH [c] (or KFH [d]), thence is [d] Ax. 10.
ICB ⊏ KFE; and consequently BI ⊏ EK [e]; whence [e] 21. 1.
also BG ($\frac{1}{2}$BI) ⊏ EH ($\frac{1}{2}$EK) or its equal BM [d]; and
therefore BA ⊏ BN, because AG and MN being
parallels, both the points M and N will fall on the
same side of AG. But BN (as the triangles NBM, [f] Hyp. and
DEH are equiangular, and have BM = EH [f]) is 7. 1.
= DE [g]: therefore BA is also greater than DE. [g] 15. 1.

THEOREM XXIII.

If, of two triangles (ABC, ABD) *ſtanding upon the ſame baſe* (AB), *the one be wholly included within the other, the two ſides* (AD, BD) *of the included one, taken together, will be leſs, and the angle* (D) *contained by them greater, reſpectively, than the two ſides* (AC, BC), *and the contained angle* (C) *of the other.*

CASE I. *If the vertex of the contained triangle be in one ſide of the other:*

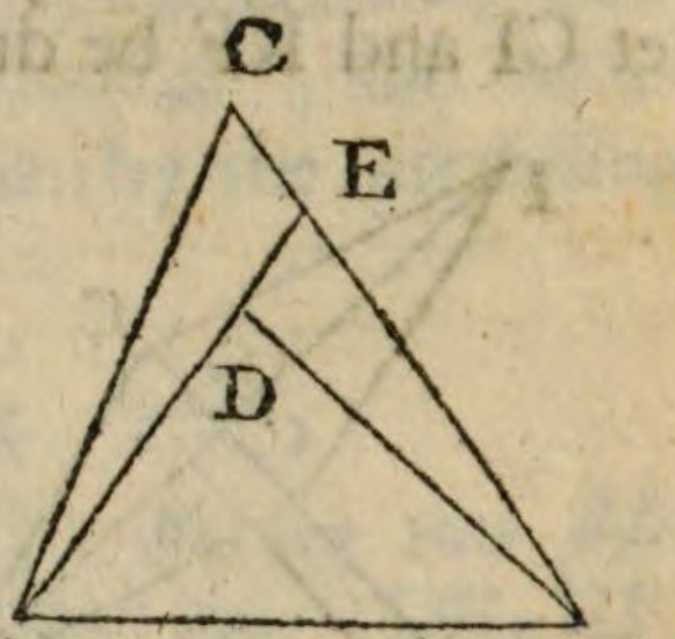


Then, AD is leſs than AC + CD [h]; whence, by adding BD common, AD + BD will alſo be leſs than AC + CD + BD [i], or than its equal AC + BC [k]. But the angle ADB is ⊏ ACB [l].

[h] 19. 1.
[i] Ax. 6.
[k] Ax. 3.
[l] Cor. 9. 1.

CASE II. *If the vertex be within the other triangle:*

Let AD be produced to meet BC in E: then *(by the preceding caſe)* the ſum of AD and BD is leſs than the ſum of AE and BE; which laſt ſum, and conſequently the former, is, again, leſs than the ſum of AC and BC. Moreover, the angle ADB ⊏ BED ⊏ C.

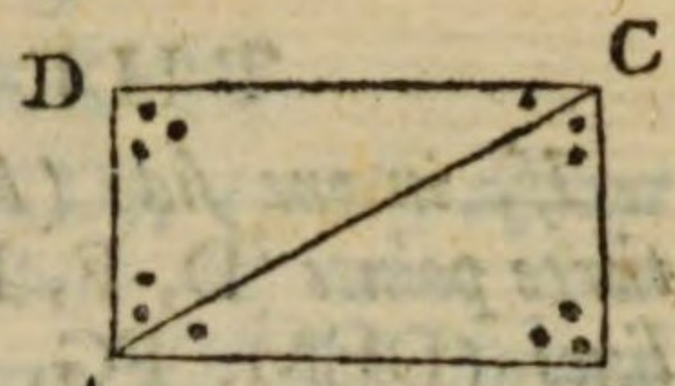


THEOREM XXIV.

The oppoſite ſides (AB, DC) *of any parallelogram* (ABCD) *are equal, as are alſo the oppoſite angles* (B, D); *and the diagonal* (AC) *divides the parallelogram into two equal parts.*

For,

For, AB, DC, and AD, BC being parallels [n], the angle BAC is = DCA [o], and BCA = DAC [o]; therefore the equiangular triangles ABC, ADC [p] having AC common, are equal in all respects [q].



[n] Def. 24.
[o] 7. 1.
[p] Cor. 1. to 10. 1.
[q] 15. 1.

COROLLARY.

Hence, if one angle (B) of a parallelogram be a right-angle, all the other three will be right ones: for D, being = B, is a right-angle; and BCD is = B, and DAB = D, *by Theor. V.*

THEOREM XXV.

Every quadrilateral (ABCD) *whose opposite sides are equal, is a parallelogram.* (See the preceding scheme.)

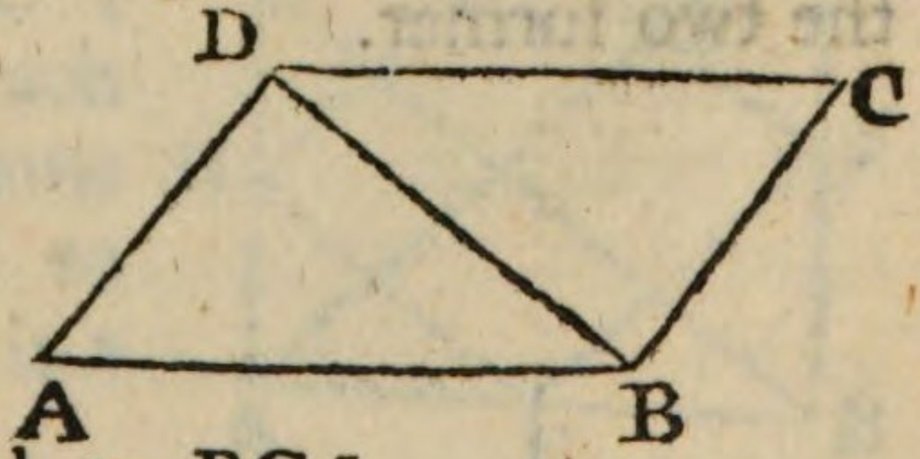
Let the diagonal AC be drawn; then the triangles ABC, ADC being mutually equilateral [r], they will also be mutually equiangular [s]; consequently AB will be parallel to DC, and AD to BC [t].

[r] Hyp.
[s] 14. 1.
[t] 8. 1.

THEOREM XXVI.

The lines (AD, BC) *joining the corresponding extremes of two equal, and parallel lines* (AB, DC) *are themselves equal and parallel.*

Let the diagonal BD be drawn. Because AB and DC are parallel [u], the angle ABD is = CDB [w]; therefore, BA being = DC [x] and BD common, the remaining sides and angles will likewise be respectively equal [y]; and consequently AD parallel to BC [z].



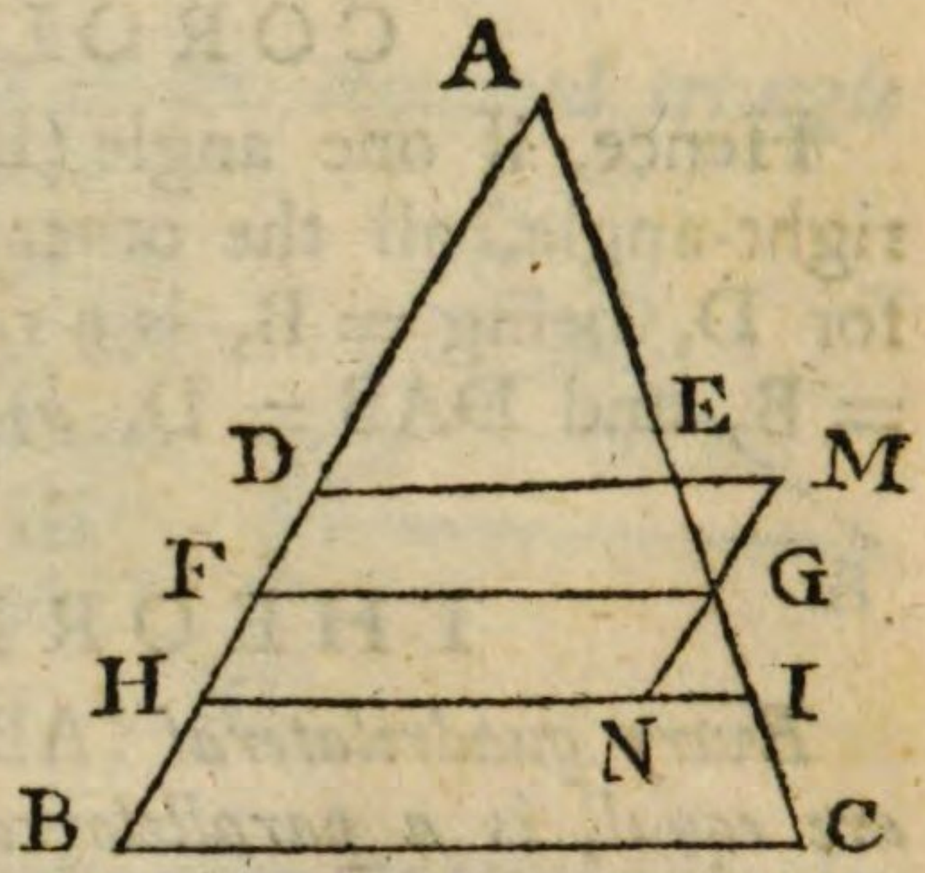
[u] Hyp.
[w] 7. 1.
[x] Hyp.
[y] Ax. 10.
[z] 8. 1.

THEO-

THEOREM XXVII.

If, in one side (AB) *of a triangle* (ABC), *from three points* (D, F, H) *at equal distances* (DF, FH), *lines* (DEM, FG, HI) *be drawn parallel to the base, the parts* (EG, GI) *of the other side* (AC) *intercepted by them, will also be equal to each other.*

Let NGM be parallel to AB, intersecting HI and DE in N and M. Then, the triangles IGN, MGE, having the
[a] 3. 1. angle IGN = EGM[a],
[b] 7. 1. ING = M[b], and GN
[c] 24. 1. (= FH[c] = FD[d]) =
[d] Hyp. GM[c], will also have
[e] 15. 1. GI = GE[e].

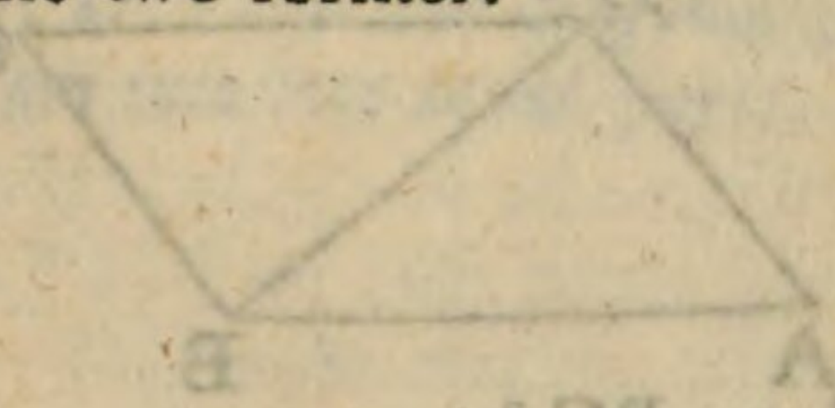


COROLLARY I.

Hence it appears, that, if one side of a triangle be divided into any number of equal parts, and from the points of division lines be drawn parallel to the base, cutting the other side, they will also divide it into the same number of equal parts.

COROLLARY II.

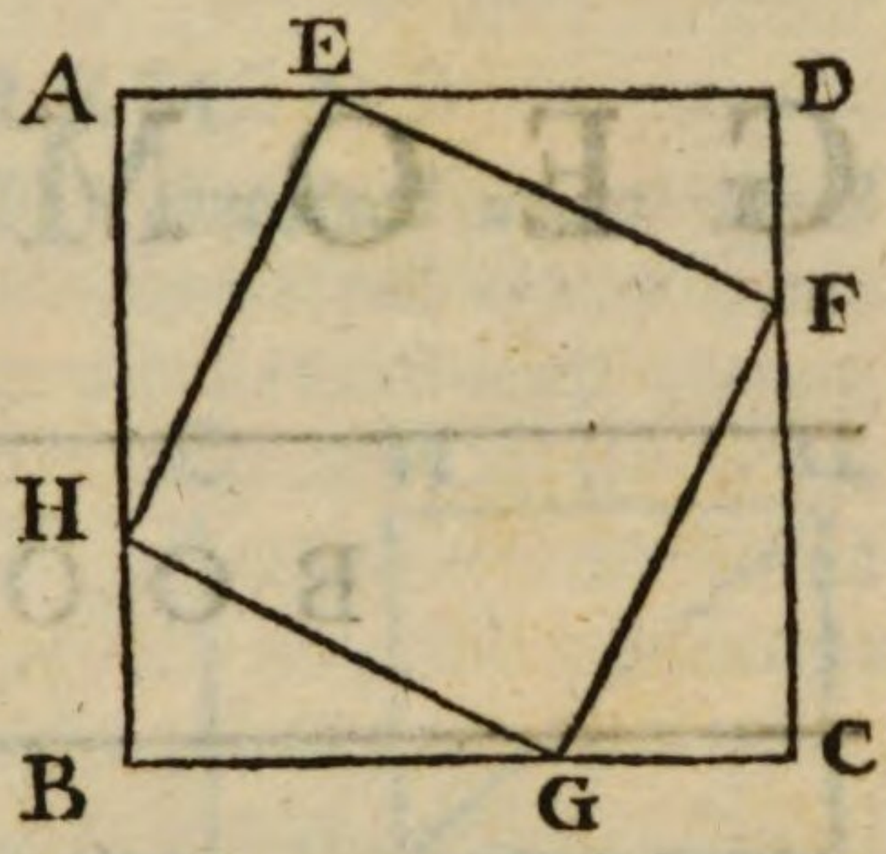
Hence, also, if two lines FG, HI, cutting the sides of a triangle, be parallel to each other, and another line DE be so drawn as to cut off FD = FH and GE = GI, this line DE will be parallel to the two former.



THEO-

THEOREM XXVIII.

If in the sides of a square (ABCD), *equally distant from the four angular points, there be taken four other points* (E, F, G, H) *the figure* (EFGH) *formed by joining those points, shall also be a square.*



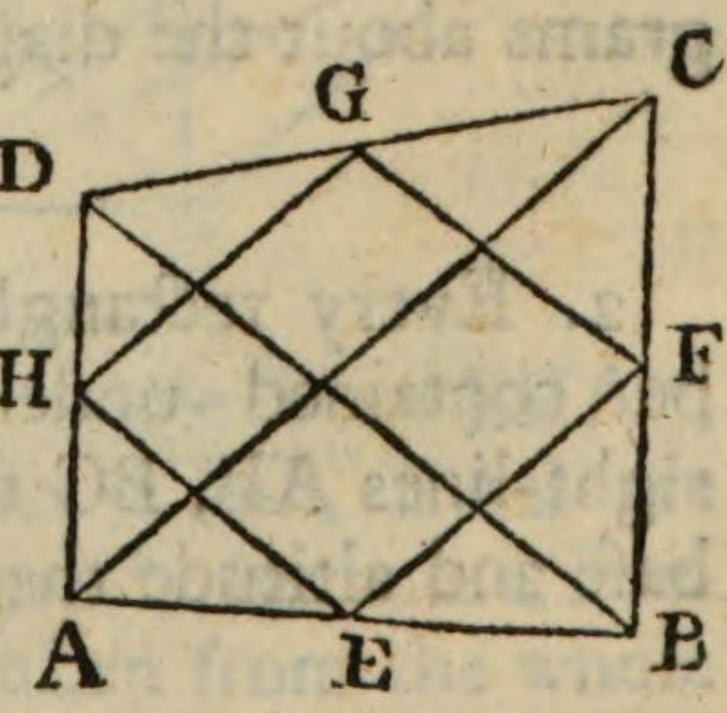
For the wholes AD, DC, CB, BA being equal [f], and also the parts AE, DF, CG, BH [g], the remaining parts ED, FC, GB, HA must consequently be equal [h]; whence, all the angles D, C, B, A being equal [i], the sides EF, FG, GH, HE will be equal likewise [k], and the angle DEF = AHE [k]. Therefore, because DEH is = A + AHE [l], if from these, the equal angles DEF, AHE be taken away, there will remain HEF = A [b] = a right-angle [f]. By the same argument (or by Theor. 25th, and the Corol. to the 24th) the other three angles will be right-angles.

[f] Def. 26.
[g] Hyp.
[h] Ax. 5.
[i] Ax. 7.
[k] Ax. 10.
[l] 9. 1.

THEOREM XXIX.

If all the sides of any quadrilateral (ABCD) *be bisected, the figure* (EFGH) *formed by joining the points of bisection, will be a parallelogram.*

Draw the diagonals AC and BD. Because EF and HG are parallel to AC [n], they are also parallel to each other [o]. After the same manner is FG parallel to EH; therefore EFGH is a parallelogram [p].



[n] Cor. 1. to 27. 1.
[o] 6. 1.
[p] Def. 24.

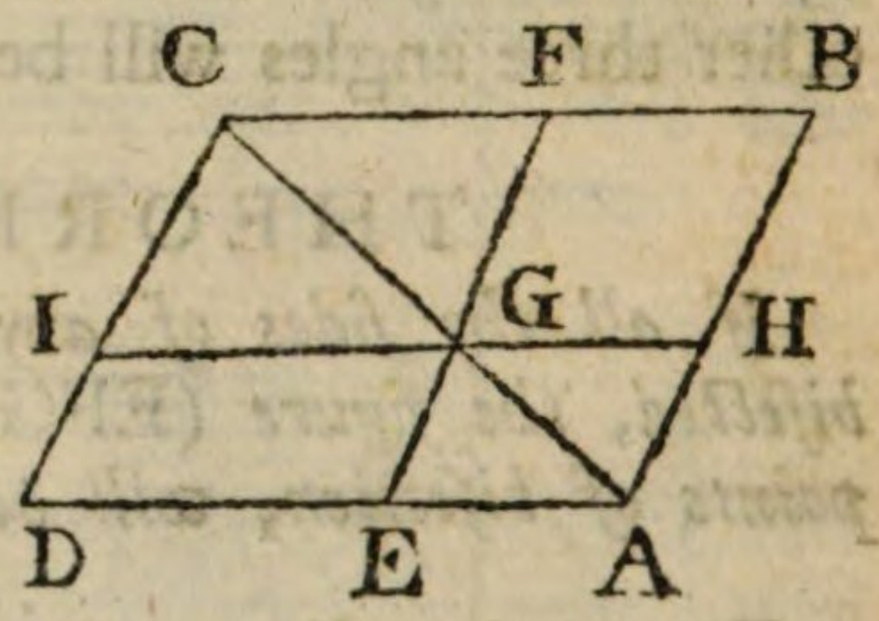
The End of the FIRST BOOK.

ELEMENTS
OF
GEOMETRY.

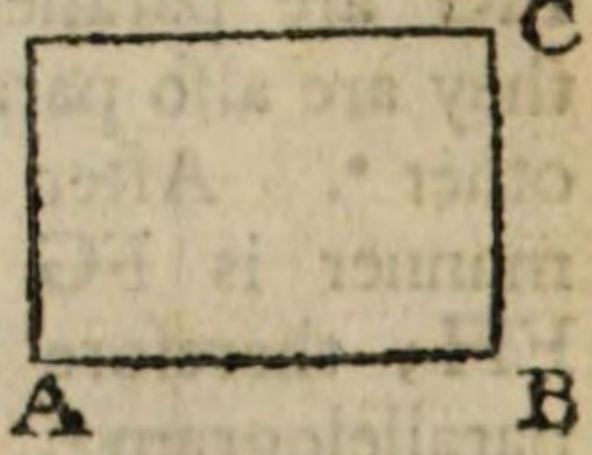
BOOK II.

DEFINITIONS.

1. IN a parallelogram ABCD, if two right-lines EF, HI, parallel to the ſides, interſecting the diagonal in the ſame point G, be drawn, dividing the parallelogram into four other parallelograms; thoſe two GD, GB through which the diagonal does not paſs, are called Complements; and the other two, HE, FI, Parallelograms about the diagonal.



2. Every rectangle is ſaid to be contained under the two right-lines AB, BC that are the baſe and altitude thereof.



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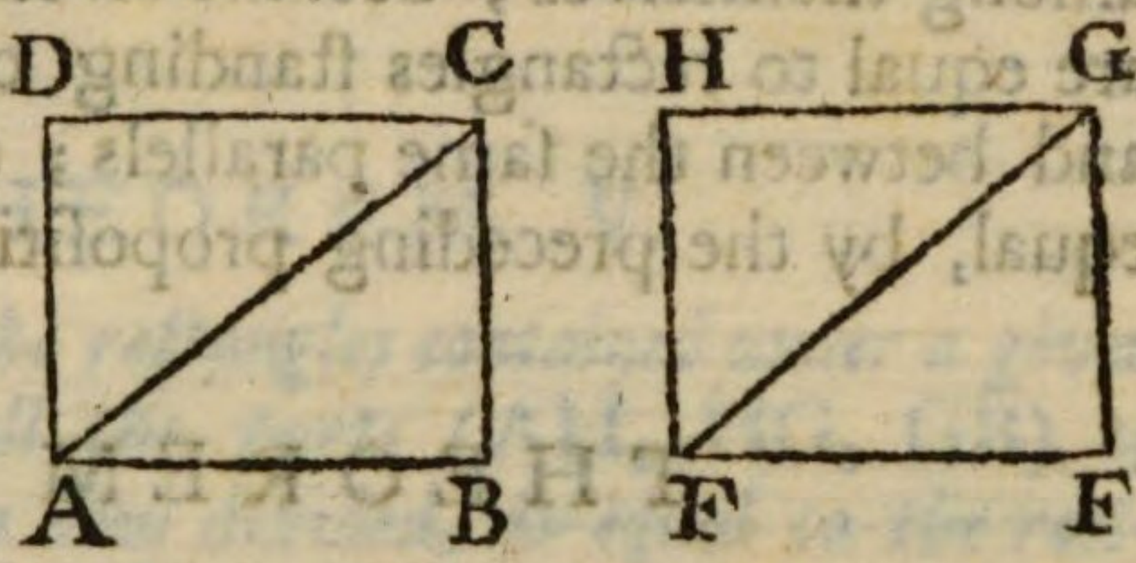
The rectangle contained under two right-lines AB *and* BC *is often, for brevity ſake, denoted by* AB×BC. *But when the figure is a ſquare, it is uſually repreſented by placing the number* 2 *over the letter, or letters expreſſing the ſide thereof: thus* AB^2 *denotes the ſquare made upon the line* AB.

THEOREM I.

The rectangles (BD, FH) *contained under equal lines, are equal.*



For, let the diagonals AC, EG be drawn: then, becauſe AB = EF, BC = FG, and B = F [a], the triangles ABC, EFG are equal [b]. And, in the very ſame manner will ADC and EHG appear to be equal. Therefore the whole rectangle ABCD is alſo equal to the whole rectangle EFGH [c].

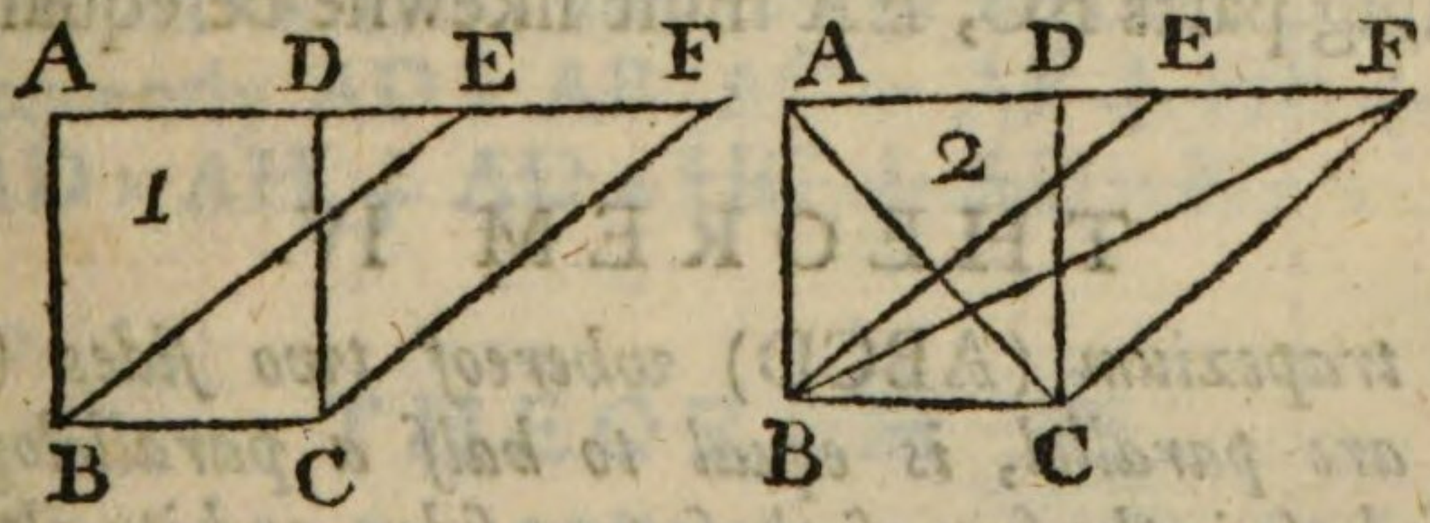
[a] Hyp.

[b] Ax. 10.

[c] Ax. 4. 1.

THEOREM II.

Parallelograms (ABCD, BCFE) *ſtanding upon the ſame baſe* (BC) *and between the ſame parallels* (BC, AF) *are equal.*



For, ſince (in *Fig.* 1.) the angle F = BEA [d], and CDF = A [d], the triangles FDC, EAB are equiangular [e]; they are alſo equal [f], becauſe CF = BE [g]: therefore, if each be taken from the whole figure ABCF, there will remain ABCD = EBCF [h].

[d] Cor. 1. to 7. 1.

[e] Cor. 1. to 10 1.

[f] 15. 1.

[g] 24. 1.

[h] Ax. 5.

COROL-

COROLLARY I.

Hence, triangles BAC, BFC *(Fig.* 2.) ſtanding upon the ſame baſe, and between the ſame parallels, are alſo equal, being the halves of their reſpective parallelograms [i].

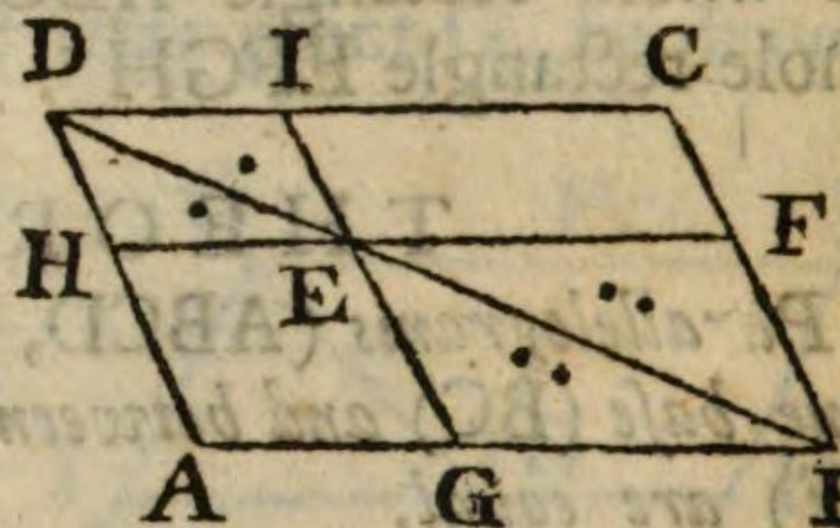
[i] 24. 1.

COROLLARY II.

Hence all parallelograms, or triangles, whatever, whoſe baſes and altitudes are equal, are equal among themſelves; becauſe all ſuch parallelograms are equal to rectangles ſtanding on the ſame baſes, and between the ſame parallels; and theſe laſt are equal, by the preceding propoſition.

THEOREM III.

The complements (EC, EA) *of any parallelogram* (AC) *are equal.*



For, the whole triangle DCB being equal to the whole triangle DAB [k], and the parts DIE, EFB reſpectively equal to the parts DHE, EGB [k], the remaining parts EC, EA muſt likewiſe be equal [l].

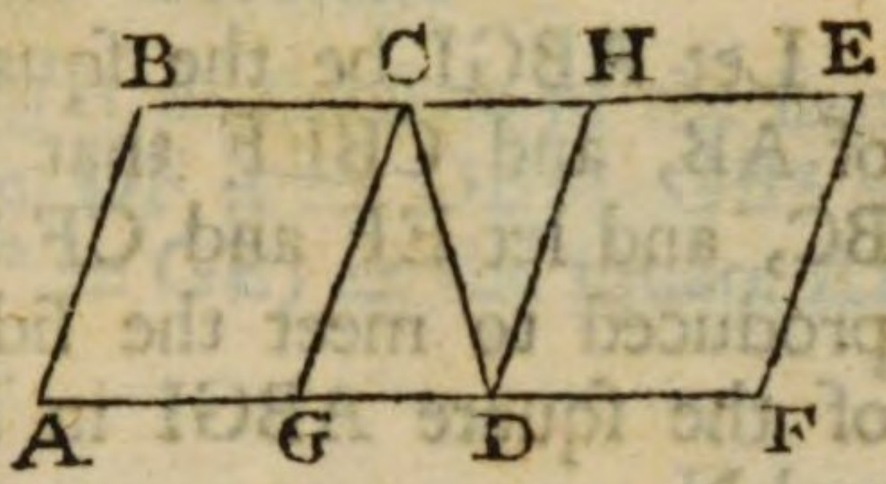
[k] 24. 1. and Def. 1.

[l] Ax. 5. 1.

THEOREM IV.

A trapezium (ABCD) *whereof two ſides* (AD, BC) *are parallel, is equal to half a parallelogram, whoſe baſe is the ſum of thoſe two ſides, and its altitude, the perpendicular diſtance between them.*

For,

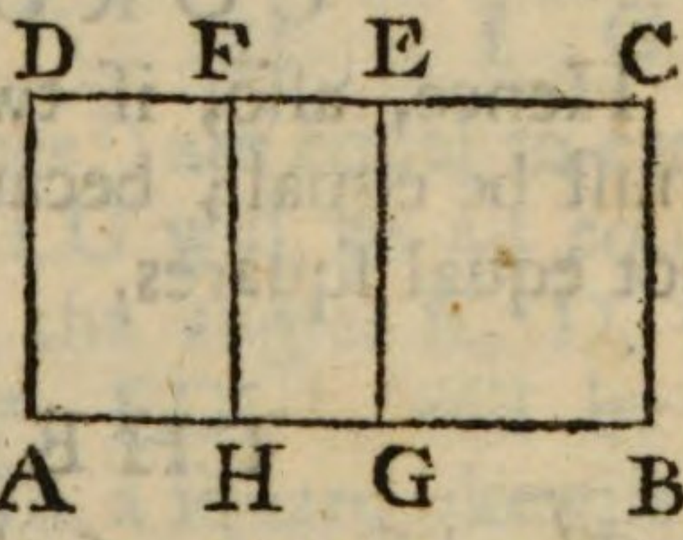


For, in AD produced, take DF = BC; and let CG, DH and FE be all parallel to AB, meeting AF and BC produced, in G, H and E. Then AE is a parallelogram of the same altitude with ABCD, having its base AF equal to the sum of AD and BC[n]: but this parallelogram, because BG = HF[o], and CGD = CHD[p], is equally divided by the line CD[q]; and so ABCD is the half thereof.

[n] Constr.
[o] Cor. 2. to 2. 2.
[p] 24. 1.
[q] Ax. 4.

THEOREM V.

The sum of all the rectangles contained under a given line (AD), *and all the parts* (AH, HG, GB) *of another* (AB), *any how divided, is equal to the rectangle contained under the two whole lines.*



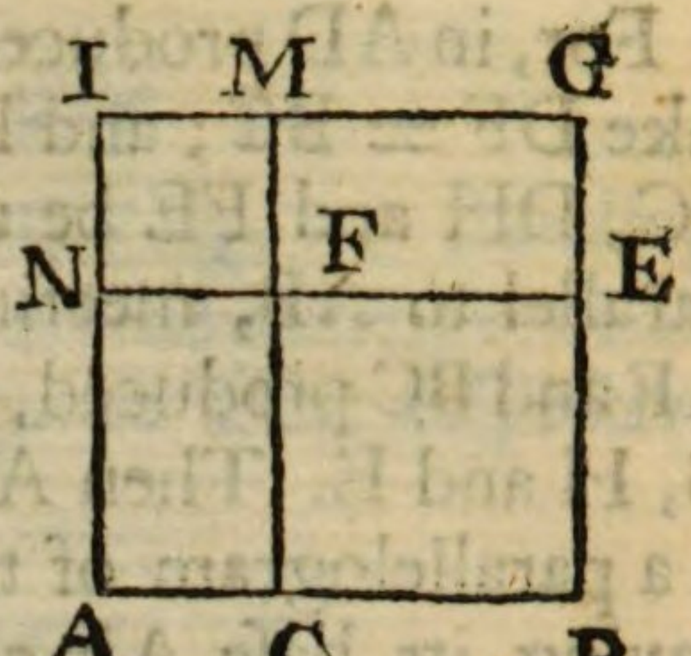
Let ABCD be the rectangle contained under the two whole lines, and let HF, GE be parallel to AD, meeting DC in F and E. Then will AF, HE, GC be rectangles[r] of the same altitude with AC[s]; therefore AF = AD × AH, HE = AD × HG, and GC = AD × BG[t]; and consequently AD × AB (AC = AF + HE + GC) = AD × AH + AD × HG + AD × BG[u].

[r] Cor. to 24. 1.
[s] 24. 1. & Ax. 1.
[t] 1. 2.
[u] Ax. 3. & 4. 1.

THEOREM VI.

If a right-line (AB) *be, any-wise, divided into two parts* (AC, BC), *the square of the whole line will be equal to the squares of both the parts, together with two rectangles under the same parts.*

Let



Let ABGI be the ſquare of AB, and CBEF that of BC, and let EF and CF be produced to meet the ſides of the ſquare ABGI in M and N.

From the equal quantities CM, EN [w] take the equal quantities CF and EF, and there remains FM = FN [x]; therefore, all the angles of the figure being right ones [y], NM is a ſquare [z] upon FN (= AC), and AF, FG are equal to two rectangles under BC and AC [a]: but AG = BF + FI + AF + FG, or $AB^2 = BC^2 + AC^2 + 2AC \times BC$ [b].

[w] 24. 1. and Defin. 26.
[x] Ax. 5. 1.
[y] Cor. to 24. 1.
[z] Def. 26.
[a] 1. 2.
[b] Ax. 3. 1.

COROLLARY I.

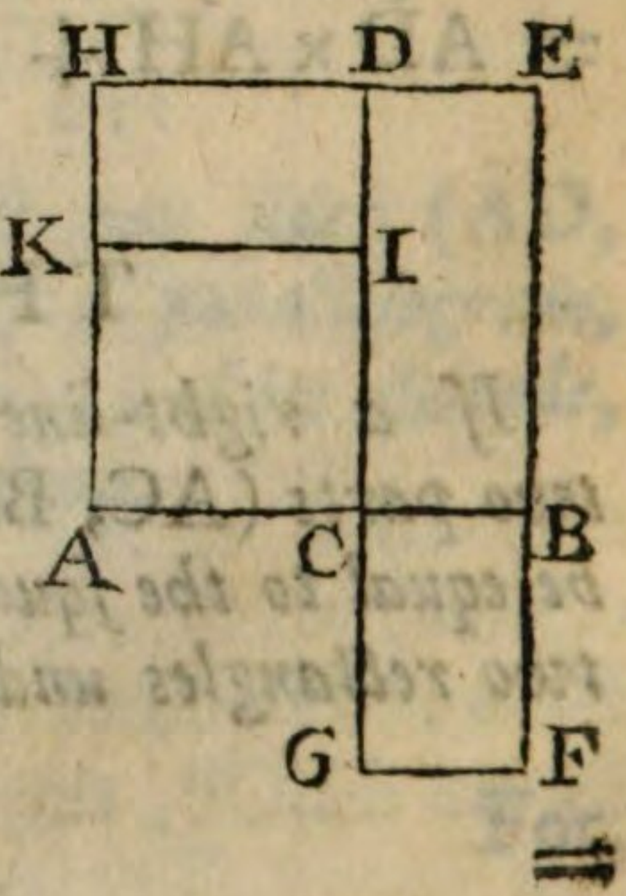
Hence, the ſquare of any line is equal to four times the ſquare of half that line.

COROLLARY II.

Hence, alſo, if two ſquares be equal, their ſides muſt be equal; becauſe unequal lines BA, BC have not equal ſquares.

THEOREM VII.

The difference of the ſquares (ABEH, ACIK) *of any two unequal lines* (AB, AC), *is equal to a rectangle under the ſum and difference of the ſame lines.*



In EB, produced, take BF = AC; let FG be drawn parallel to EH, and let CI be produced both ways, to meet EH and FG in D and G. It is evident that DF is a rectangle [c], whoſe baſe GF (= CB [d]) = the difference of the given lines AB, AC; and whoſe altitude FE (becauſe BE

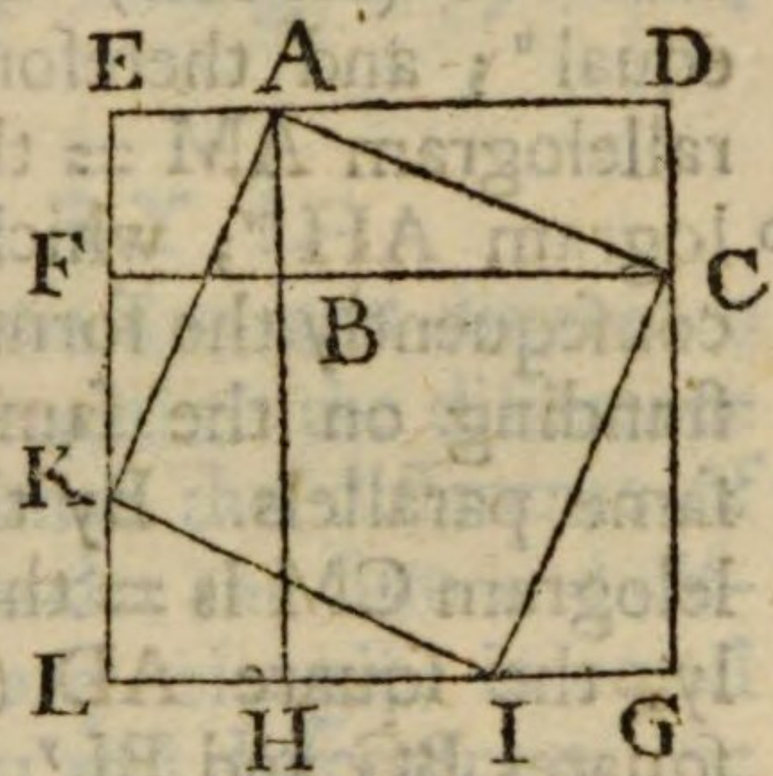
[c] Cor. 24. 1.
[d] 24. 1.

= BA [c], and BF = AC [f]) is = the sum of the same lines: but this rectangle DF is = DB + GB [g] = DB + DK (because DK [h] = GB) = the square AE — the square AI.

[c] Def. 24. [f] Hyp. [g] Ax. 3. [h] 1. 2.

THEOREM VIII.

The square made upon the side (AC) *subtending the right-angle of a plane triangle* (ABC), *is equal to both the squares* (BE, BG) *made upon the sides* (AB, BC) *containing that angle.*



Let the sides of the squares BE, BG be produced to meet each other in L and D; in which take KL and IG each equal to AE (or AB); and let CI, IK, and KA be drawn.

Since ABH and FBC (which are continued right-lines [i]) are equal to each other [k], EL, DG, ED, and LG will be all equal among themselves [l]; and so, the angles E, D, G and L being all right-ones [m], EDGL will be a square, and consequently ACIK a square likewise [n]. Now, if from the square DL, the four equal [o] triangles ADC, CGI, ILK, and KEA be taken away, there will remain the square AI: and, if from the same DL, the two equal [p] parallelograms DB, BL (which are equal to the said four triangles, because DB = two of them [l]) be taken away; then there will remain the two squares BE and BG. Consequently the square AI is = the two squares BE and BG [q].

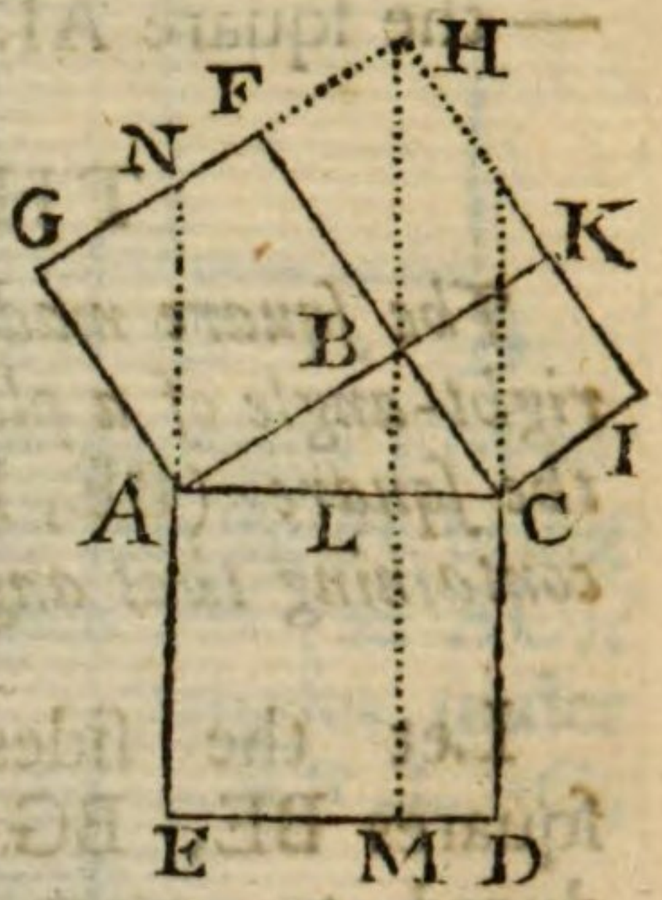
[i] 2. 1. [k] Ax. 4. [l] 24. 1. [m] Hyp. & 5. 1. [n] 28. 1. [o] Ax. 10. [p] 1. 2. [q] Ax. 5.

The same demonstrated otherwise.

Let AD be the square on the hypothenuse AC, and BG, BI the two squares on the sides AB and

BC: let MBH be parallel to AE, meeting GF (produced) in H; and let EA be produced to meet GH in N.

[r] Ax. 7. If from the equal [r] angles GAB, CAN, the angle NAB, common to both, be taken away, there will remain NAG [s] Ax. 5. = BAC [s]; whence, as the angle G is also = ABC [t], and the [t] Def. 26. side AG = AB [t], the sides AN and AC (= AE) are likewise [u] 15. 1. equal [u]; and therefore the parallelogram AM = the parallelogram AH [w]; which last, and [w] Cor. to 2. 2. consequently the former, is equal to the square BG [x] [x] 2. 2. standing on the same base AB, and between the same parallels. By the same argument, the parallelogram CM is = the square BI: and, consequently, the square AD (= AM + CM) = both the [y] Ax. 4. squares BG and BI [y].



COROLLARY.

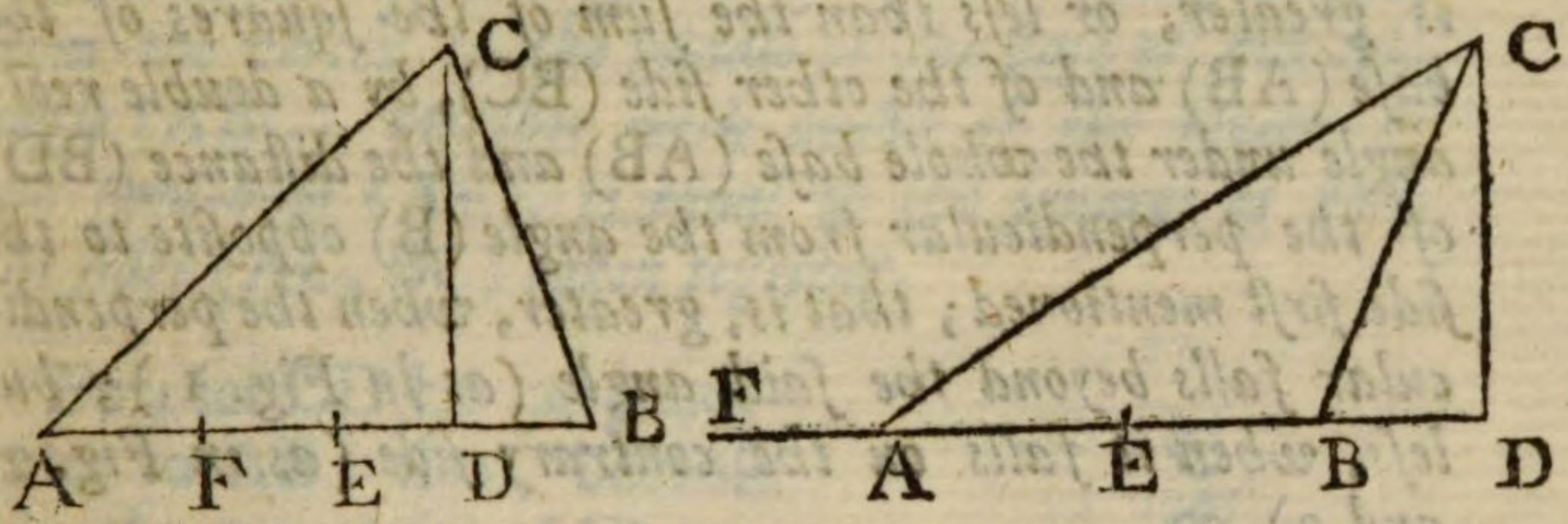
Hence, the square upon either of the sides including the right-angle, is equal to the difference of the squares of the hypothenuse and the other [z] Ax. 5. side [z]; or, equal to a rectangle contained under the sum and difference of the hypothenuse and the other [a] 7. 2. side [a].

THEOREM IX.

The difference of the squares of the two sides (AC, BC) *of any triangle* (ABC) *is equal to the difference of the squares of the two lines, or distances* (AD, BD) *included between the extremes of the base* (AB) *and the perpendicular* (CD) *of the triangle.*

For, since $AC^2 = DC^2 + AD^2$, and $BC^2 = DC^2 + BD^2$ *(by the precedent)*, it is evident that the difference

difference of AC^2 and BC^2 will be equal to the difference between $DC^2 + AD^2$ and [b] $DC^2 + BD^2$, or [b] Ax. 5.



between AD^2 and [c] BD^2, by taking away DC^2, [c] Ax. 6. common, from both.

COROLLARY I.

Since the rectangle under the sum and difference of any two unequal lines, is equal to the difference of their squares [d], it follows, that the difference of [d] 7. 2. the squares (or the rectangle under the sum and difference) of the two sides of any triangle, is equal to the rectangle under the sum and difference of the distances included between the perpendicular and the two extremes of the base.

COROLLARY II.

It follows, moreover, that the difference of the squares (or the rectangle under the sum and difference) of the two sides of a triangle, is equal to twice a rectangle under the whole base, and the distance of the perpendicular from the middle of the base.

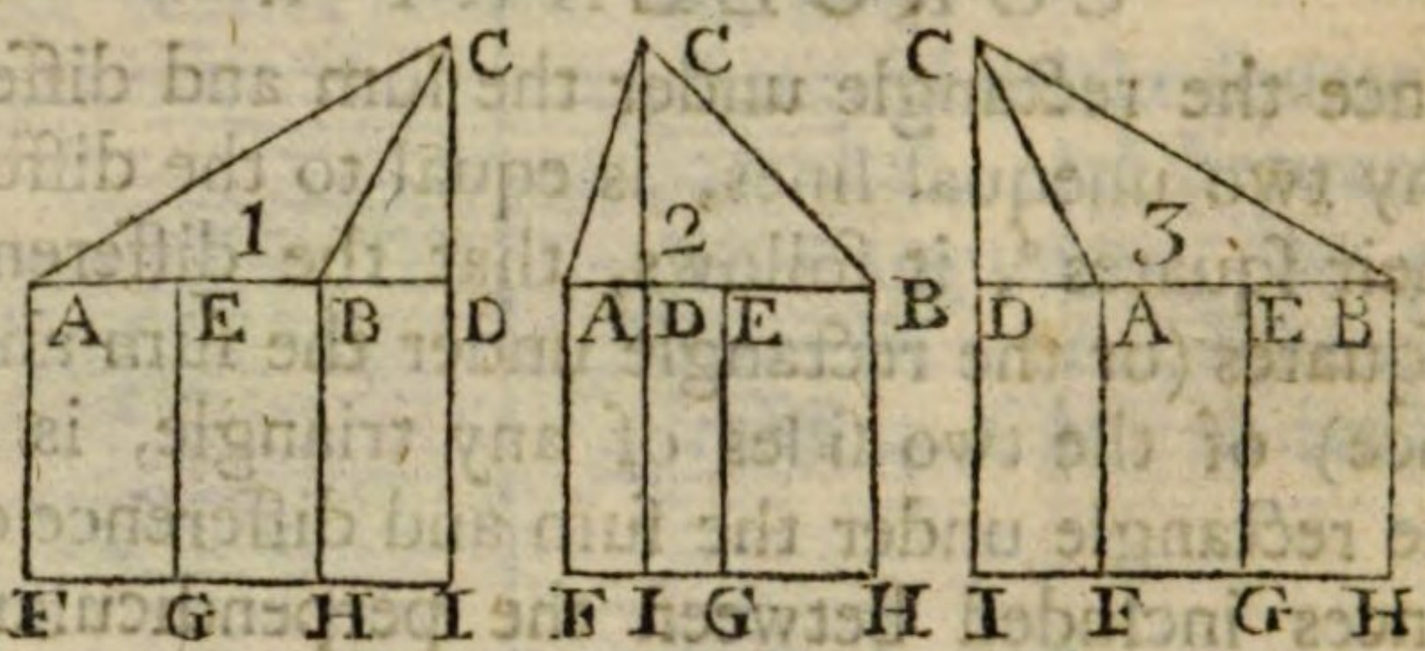
For, let E be the middle of the base, and let EF be made $=$ ED; then AF being $=$ BD [e], the excess of AD above BD (or AF) will (*in Fig.* 1.) be $=$ DF $=$ 2DE; therefore the rectangle under the sum and difference of AD and BD ($=$ [g] $AC^2 -$ [g] 9. 2. BC^2) is $=$ AB $\times$ 2DE. Again (*in Fig.* 2.) AD $+$ BD being $=$ AD $+$ AF [h] $=$ FD $=$ 2ED, and [h] Ax. 4. AD $-$ BD $=$ AB, we have, also, in this case [g], $AC^2 - BC^2 =$ AB $\times$ 2DE.

[e] Ax. 5.

THEOREM X.

The ſquare of one ſide (AC) *of a triangle* (ABC) *is greater, or leſs than the ſum of the ſquares of the baſe* (AB) *and of the other ſide* (BC), *by a double rectangle under the whole baſe* (AB) *and the diſtance* (BD) *of the perpendicular from the angle* (B) *oppoſite to the ſide firſt mentioned; that is, greater, when the perpendicular falls beyond the ſaid angle (as in Fig.* 1.); *but leſs, when it falls on the contrary ſide (as in Fig.* 2. *and* 3).

Let the ſquare ABHF, on the baſe AB, be di-
[i] 1. 2. vided into two equal[i] rectangles EF and EH by



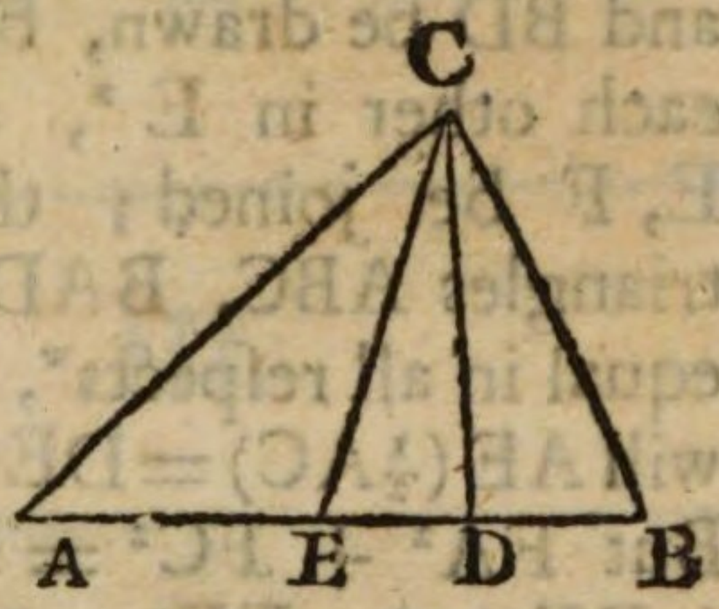
the line EG, biſecting AB in E; and let the perpendicular CD be continued out to meet FH (produced) in I.

In Fig. 1. $AC^2 - BC^2 =$ twice the rectangle
[l] Cor. 2. to 9. 2. $EI^{l} = 2EH + 2BI^{m} = AH\ (AB^2) + 2BI\ (2AB \times BD)$; therefore, if from the firſt and laſt
[m] Ax. 3. of theſe equal quantities, AB^2 be taken away, then
[n] Ax. 5. AC^2 *leſs* both BC^2 and $AB^2 = 2AB \times BD^{n}$.

In Fig. 2 *and* 3. $BC^2 - AC^2 = 2EI^{l} = 2BI - 2BG^{m} = 2AB \times BD - AB^2$; and ſo, by adding AB^2 to the firſt and laſt of theſe equal quantities,
[o] Ax. 4. we have here $AB^2 + BC^2 - AC^2 = 2AB \times BD^{o}$.

THEOREM XI.

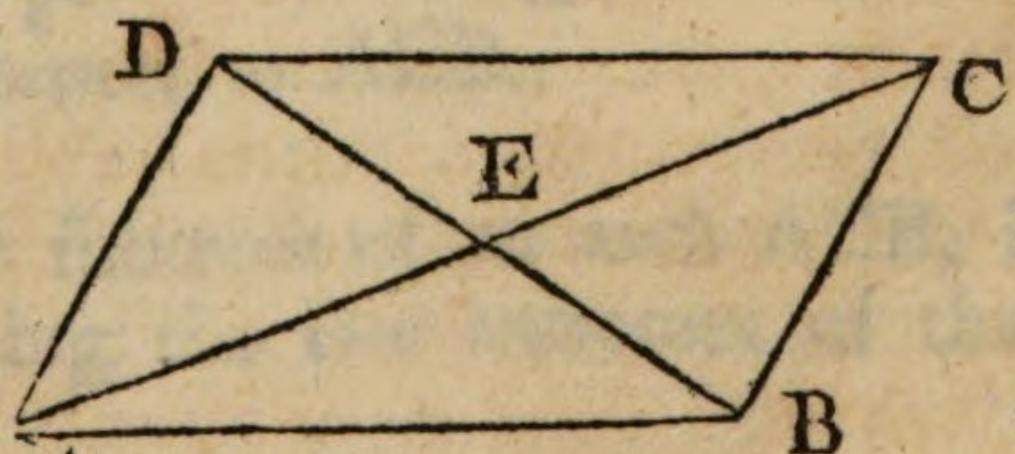
The double of the ſquare of a line (CE) *drawn from the vertex to the middle of the baſe of any triangle* (ABC), *together with double of the ſquare of the ſemi-baſe* (AE), *is equal to the ſquares of both the ſides* (AC, BC) *taken together.*



For, let CD be perpendicular to AB: then, becauſe *(by the precedent)* AC^2 exceeds the ſum of the two ſquares AE^2 and CE^2 (or BE^2 and CE^2) by the double rectangle $2AE \times ED$ (or $2BE \times ED$); and becauſe BC^2 is leſs than the ſame ſum by the ſame double rectangle; it is manifeſt that both AC^2 and BC^2 together, muſt be equal to that ſum twice taken; the exceſs on the one part making up the defect on the other.

THEOREM XII.

The two diagonals (AEC, BED) *of a parallelogram* (ABCD) *biſect each other; and the ſum of their ſquares is equal to the ſum of the ſquares of all the four ſides of the parallelogram.*

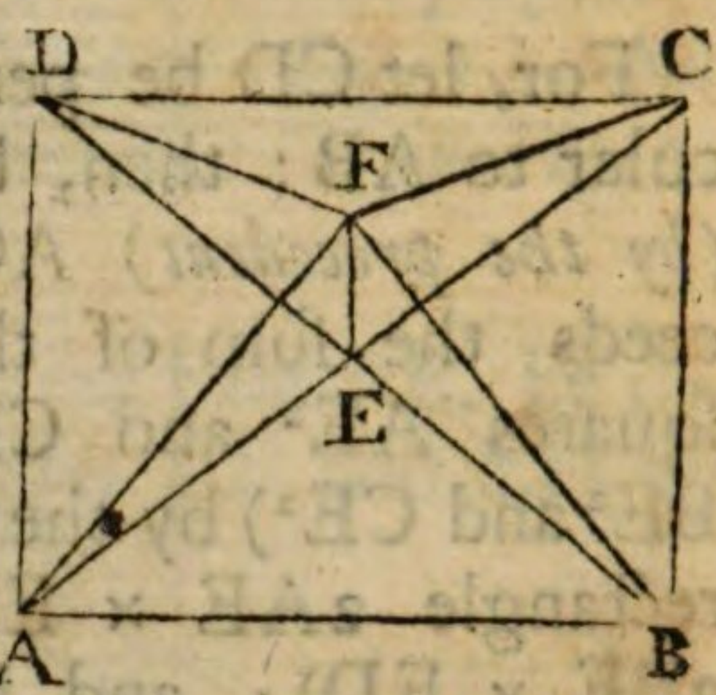


For, the triangles AEB, DEC being equiangular [p], and having AB = DC [q], will alſo have AE = CE, and BE = DE [r]. Moreover, becauſe $2AE^2 + 2ED^2 =$ [s] $AD^2 + CD^2$, by taking the double of theſe, we have $4AE^2$ ([t] AC^2) $+ 4ED^2$ (DB^2) = [u] $AD^2 + BC^2 + CD^2 + AB^2$.

[p] 3. and 7. 1.
[q] 24. 1.
[r] 15. 1.
[s] 11. 2.
[t] Cor. 1. to 6. 2.
[u] Ax. 4. and 24. 1.

THEOREM XIII.

If from any point (F), *to the four angles of a rectangle* (ABCD) *four lines be drawn; the ſums of the ſquares of thoſe drawn to the oppoſite angles will be equal* (*I ſay, that* $FA^2 + FC^2 = FB^2 + FD^2$).



For, let the diagonals AC and BD be drawn, biſecting each other in E [x], and let E, F be joined; then the triangles ABC, BAD being equal in all reſpects [w], thence will AE ($\frac{1}{2}$AC) = DE ($\frac{1}{2}$DB). But $FA^2 + FC^2 =$ [y] $2AE^2$ $(2DE^2) + 2EF^2 =$ [y] $FB^2 + FD^2$.

[x] 12. 2.
[w] 24. 1. and Ax. 10.
[y] 11. 2.

The End of the SECOND BOOK.

ELEMENTS

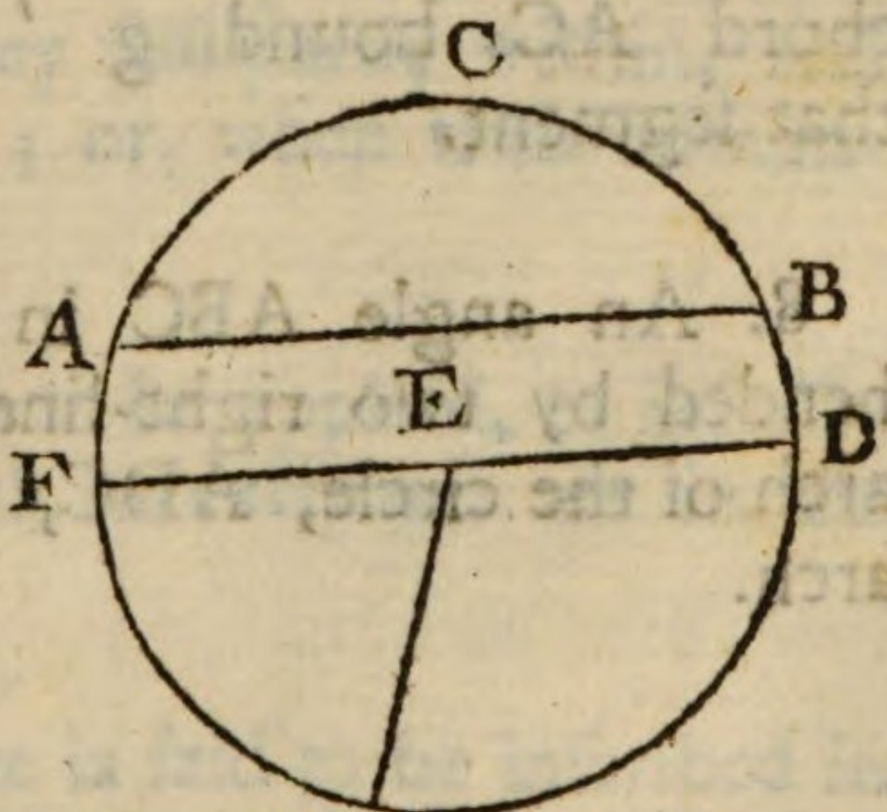
OF

GEOMETRY.

BOOK III.

DEFINITIONS.

1. ANY right-line FD, passing through E the center of a circle, and terminating in the circumference at both ends, is called a Diameter.



2. An arch of a circle, is any portion of the periphery, or circumference, as ACB.

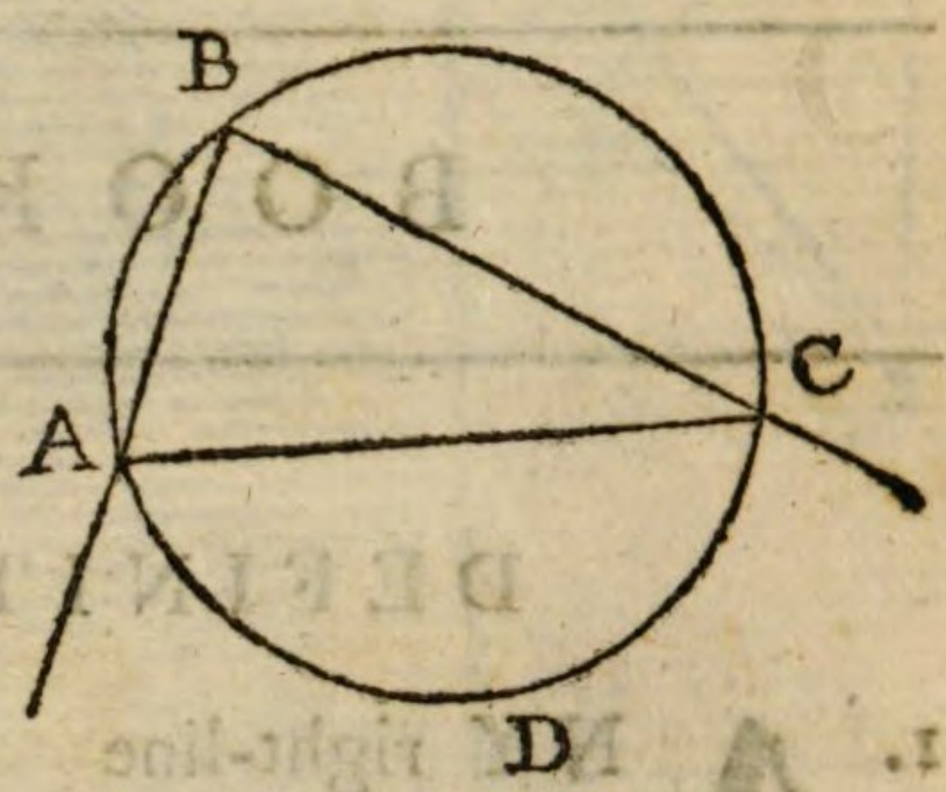
3. The chord, or ſubtenſe of an arch ACB, is a right-line AB joining the two extremes of that arch.

4. A ſemi-circle is a figure contained under any diameter and either part of the circumference cut off by that diameter.

5. A ſegment of a circle is a figure contained under an arch ACB and its chord AB.

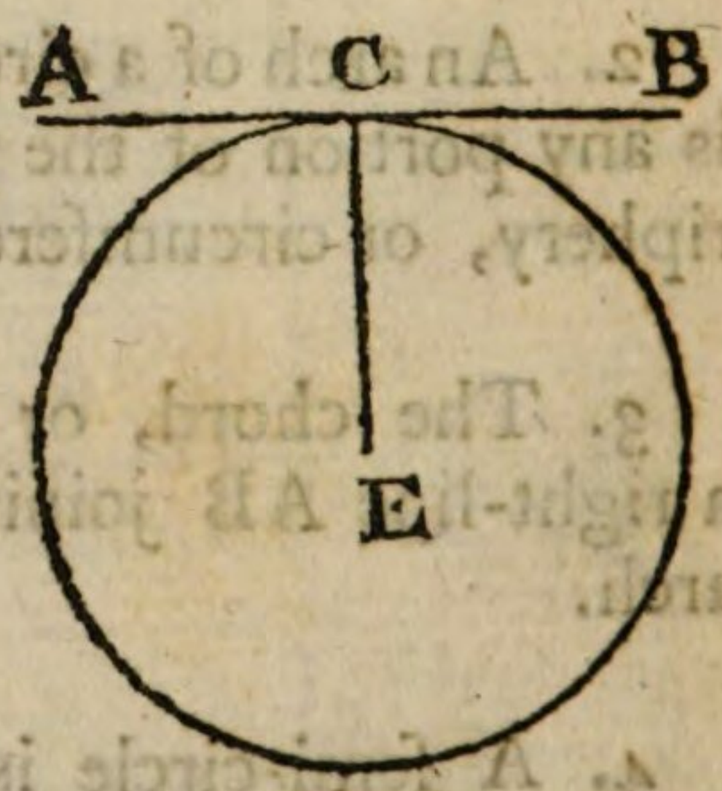
6. A Sector of a circle is a figure contained under two right-lines EF, EG, drawn from the center to the circumference, and the arch FG included betwixt them. When the two lines EF, EG ſtand perpendicular to each other, then the Sector is called a Quadrant.

7. An angle ABC is ſaid to be in a ſegment of a circle ABC, when, being in the periphery thereof, the right-lines BA, BC by which it is formed, paſs through the extremes of the chord AC bounding that ſegment.



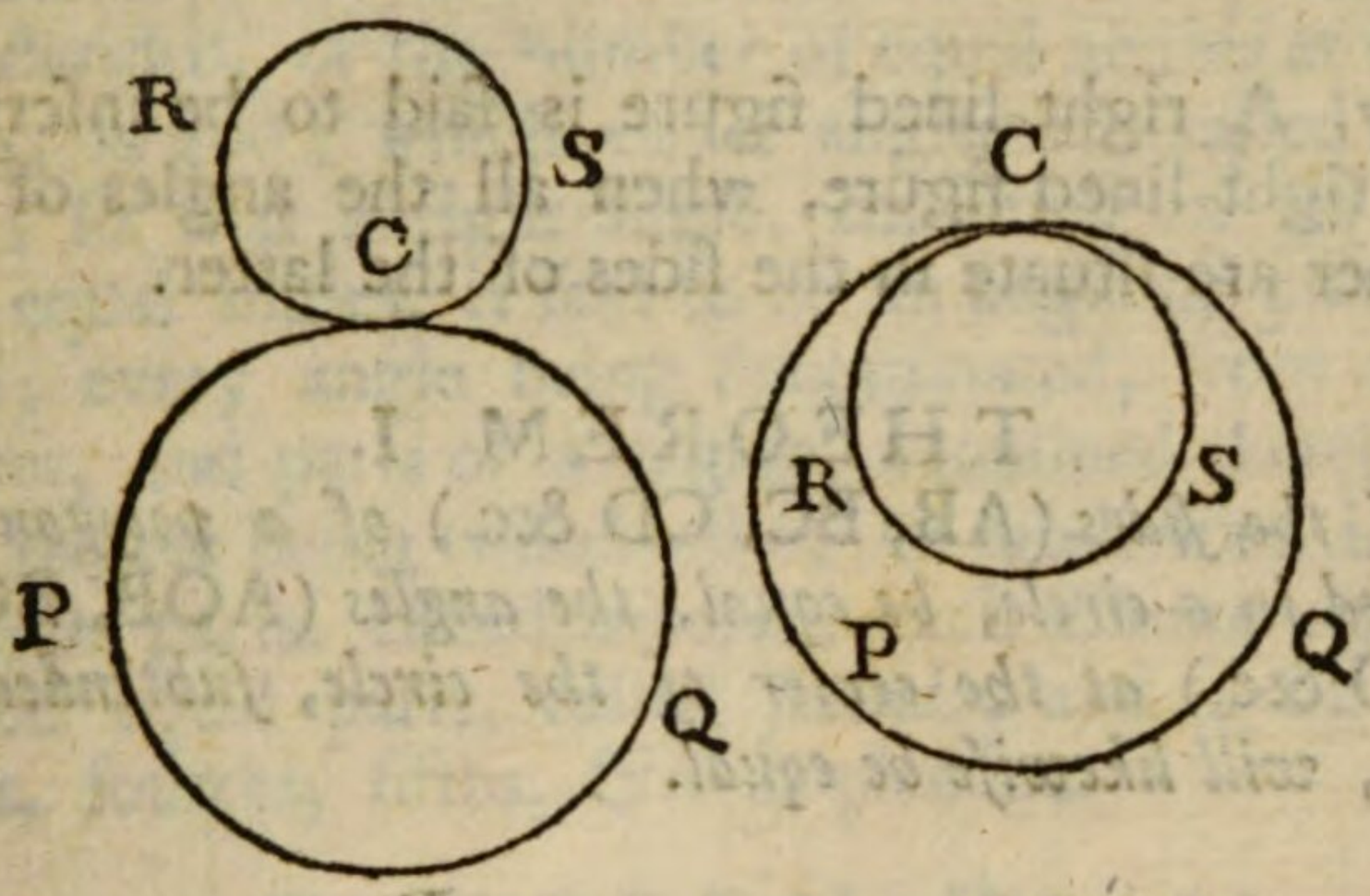
8. An angle ABC in the periphery, comprehended by two right-lines BA, BC, including an arch of the circle, ADC, is ſaid to ſtand upon that arch.

9. A right-line AB is ſaid to touch a circle, when, paſſing through a point (C) in the circumference thereof, it cutteth off no part of the circle.



10. Two

10. Two circles (PCQ, RCS) are ſaid to touch each other, when the circumferences of both paſs



through one point (C) and yet do not cut each other.

11. Two circles, in the ſame plane, are ſaid to cut one another, when they fall partly within, and partly without each other; or, when their circumferences cut each other.

12. A right-line is ſaid to be applied, or inſcribed in a circle, when both its extremes are in the periphery thereof.

13. A right-lined figure is ſaid to be inſcribed in a circle, when all its angles are in the circumference of the circle.

14. A circle is ſaid to be deſcribed about a right-lined figure, when the periphery of the circle paſſes through all the angles of that figure.

15. A right-lined figure is ſaid to be deſcribed about a circle, when all the ſides thereof touch the circle.

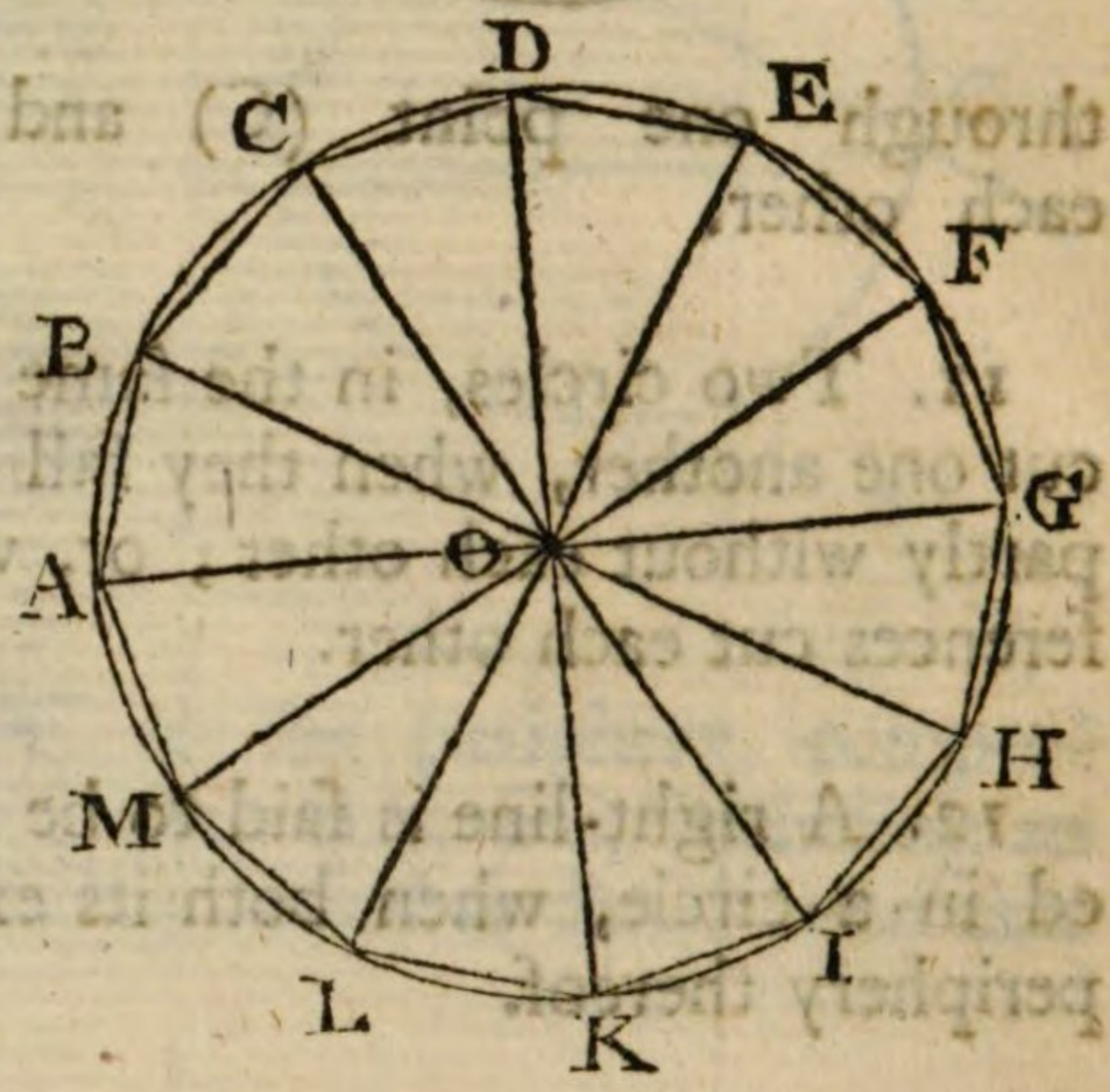
16. A

16. A circle is ſaid to be inſcribed in a right-lined figure, when it is touched by all the ſides of the right-lined figure.

17. A right-lined figure is ſaid to be inſcribed in a right-lined figure, when all the angles of the former are ſituate in the ſides of the latter.

THEOREM I.

If the ſides (AB, BC, CD &c.) *of a polygon inſcribed in a circle, be equal, the angles* (AOB, BOC, COD &c.) *at the center of the circle, ſubtended by them, will likewiſe be equal.*



For AO, BO, CO &c. being equal to each other [a], as well as AB, BC, CD &c. the triangles AOB, BOC, COD, are mutually equilateral; and therefore have all the angles AOB, BOC &c. equal to each other [b].

[a] Def. 33. of 1.

[b] 14. 1.

SCHOLIUM.

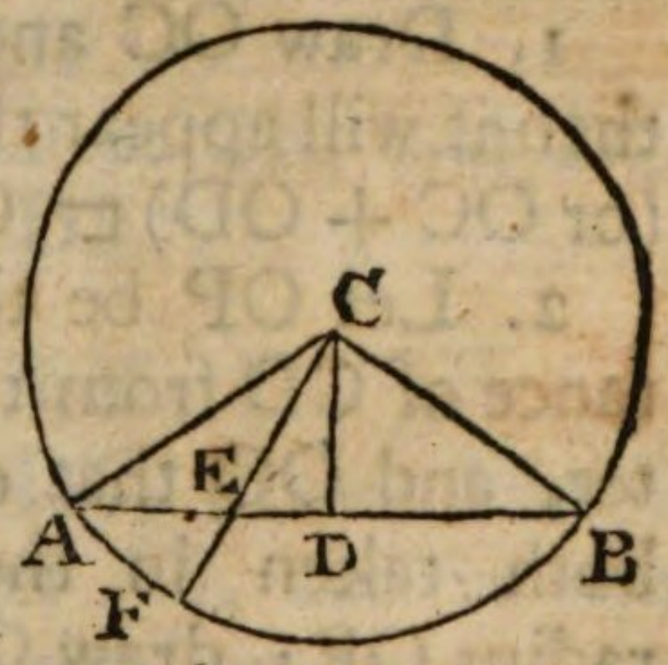
On this propoſition depends the diviſion of mathematical inſtruments for taking and meaſuring of angles. For, if, by repeated trials, or any other means, the circumference of a circle deſcribed about a center O, be divided into any number of parts AB, BC, CD &c. ſo that the chords be equal; then it is evident, from hence, that all the angles AOB, BOC, COD &c. which make up the four right-angles AOD, DOG, GOK, KOA at the center, will alſo be equal to each other, let

the

the radius OA of the *inſtrument* be what it will.—In the diviſion of the circle for practical uſes, the number of parts into which the circumference is thus divided, or the number of equal angles at the center, is 360; which equal angles are called *degrees*; ſo that a right-angle, conſiſting of 90 of theſe equal angles, is ſaid to be an angle of 90 degrees; every angle being denominated, from the degrees, and parts of a degree, contained therein; each degree being conceived to be ſubdivided into 60 equal parts, called *minutes*; each minute again into 60 equal parts, called *ſeconds*; and ſo on to thirds, fourths, fifths, *&c.* at pleaſure.

THEOREM II.

Any chord (AB) *of a circle, falls wholly within the ſame: And a perpendicular* (CD) *let fall thereon, from the center of the circle, will divide it into two equal parts.*



Let C, A, and C, B be joined; and thro' any point E in the chord AB, let the right-line CEF be drawn, meeting the circumference in F.

It is evident, becauſe CA = CB [c], that theſe equal lines are on different ſides of the perpendicular CD [d]: And ſo, CE being ⊐ CA or CF [d], the point E (take it where you will in the line AB) and conſequently the line AB itſelf, will fall within the circle [e]. Moreover, becauſe the triangles ACD, BCD have CA = CB and CD common, thence will AD be alſo = BD [f].

[c] Def. 33. 1.

[d] 20. 1.

[e] Ax. 2.

[f] 16. 1.

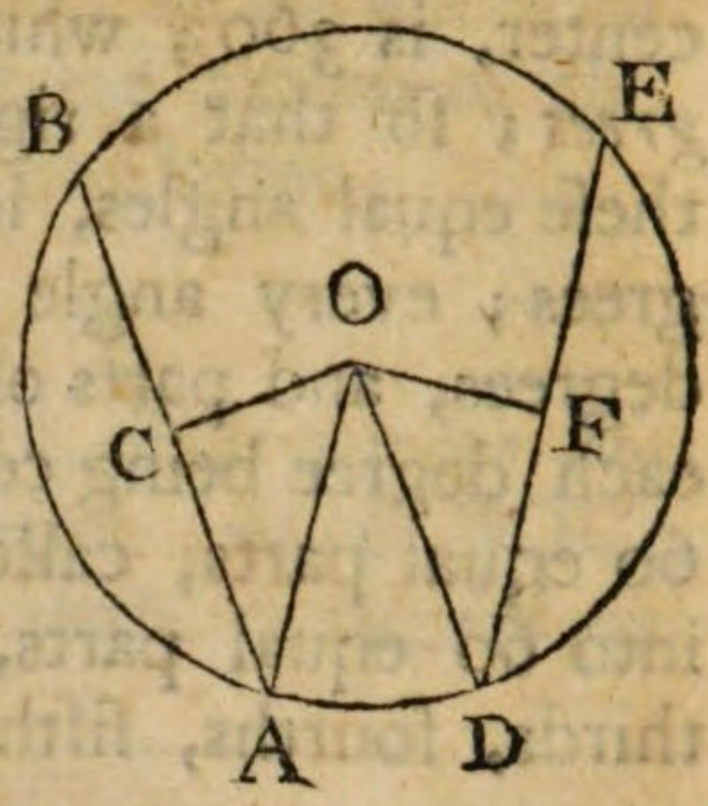
COROLLARY.

Hence a line biſecting any chord at right-angles, paſſes thro' the center of the circle.

THEO-

THEOREM III.

Any two chords (AB, DE) *equally distant from the center* (O) *of a circle, are equal to each other.*

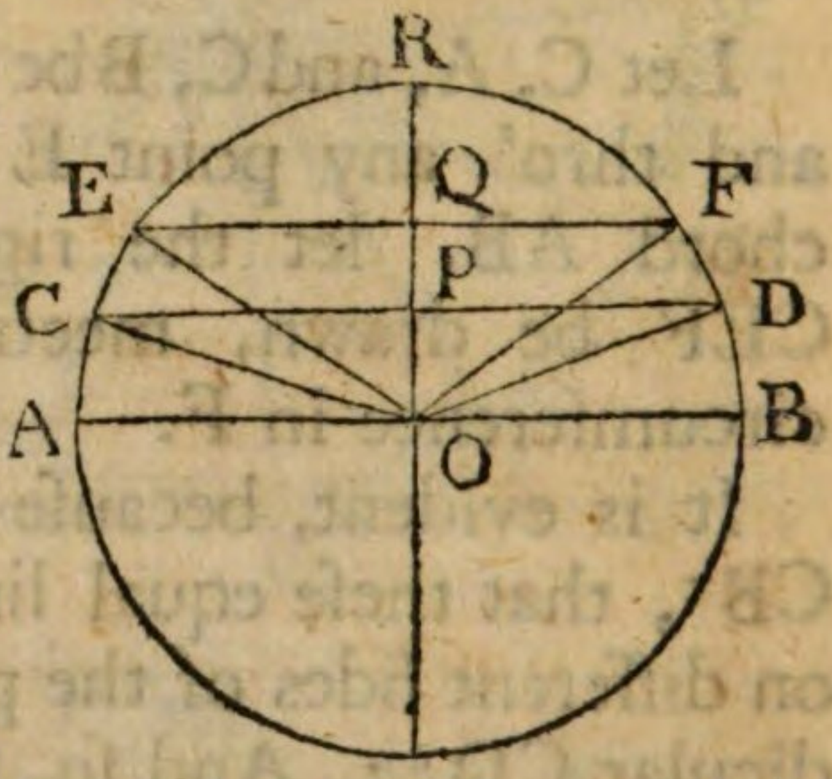


Let the perpendiculars OF, OC be drawn, and let O, D and O, A be joined. Because $OF = OC$ [e], $OD = OA$ [f], and F and C are both right-angles [g], therefore is $DF = AC$ [h], and consequently $DE = 2DF$ [i] $= 2AC$ [k] $= AB$ [i].

[e] Hyp.
[f] Def. 33. of 1.
[g] Constr.
[h] 16. 1.
[i] 2. 3.
[k] Ax. 4. 1.

THEOREM IV.

In a circle (AEFB) *the greatest line* (AB) *is the diameter; and, of all others terminating in the circumference, that* (CD) *which is nearest the center* (O), *is greater than any other* (EF) *further from it.*



1. Draw OC and OD; then it will appear that AB (or $OC + OD$) $\sqsubset$ CD [m].

[m] 19. 1.

2. Let OP be the distance of CD from the center, and OQ that of EF, both taken in the same radius OR; draw OE and OF. Because the triangles DOC, OFE, have two sides equal each to each [n], and have the contained angle DOC $\sqsubset$ the contained angle FOE [o]; therefore, also, will the base DC be greater than the base FE [p]; and, consequently, greater than any other chord at the same distance with EF [q].

[n] Def. 33. 1.
[o] Ax. 2.
[p] 21. 1.
[q] 3. 3.

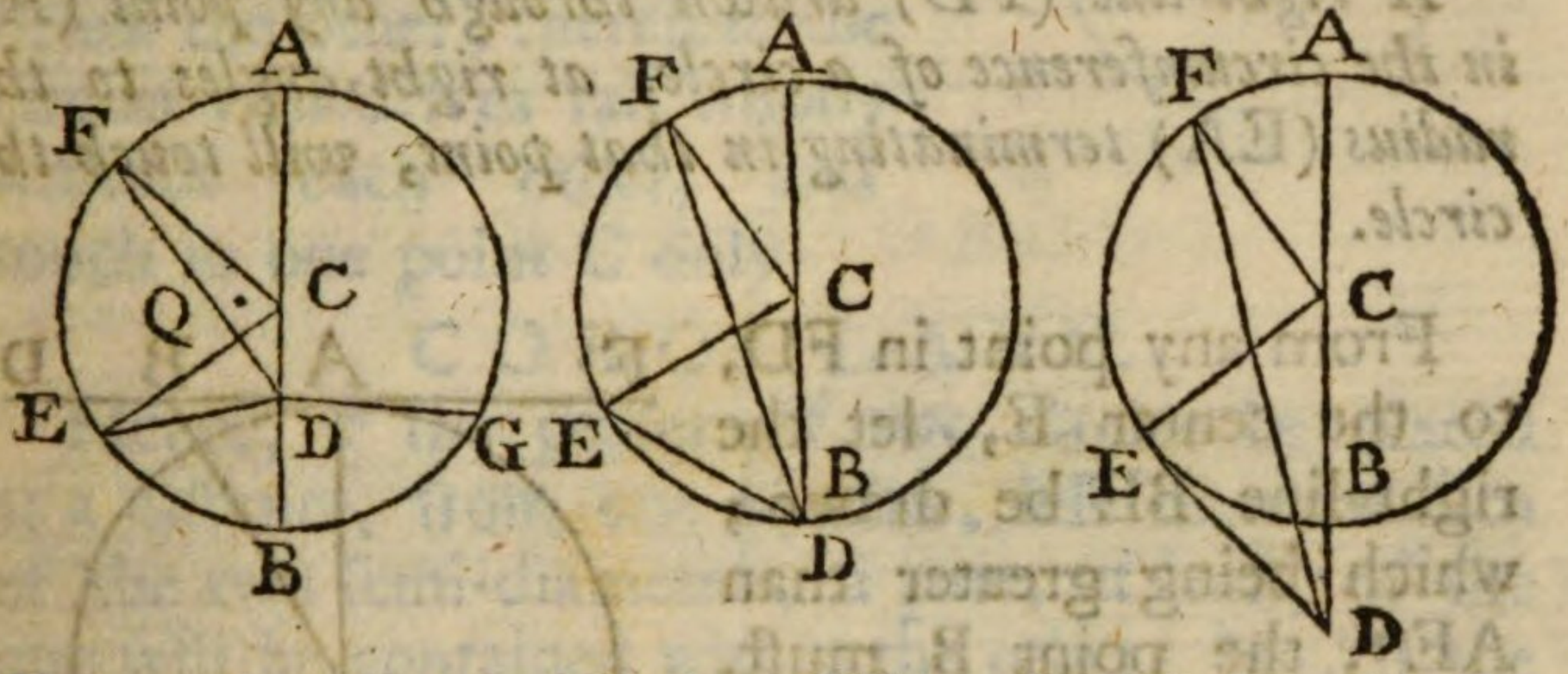
COROLLARY.

Hence, a right-line greater than the diameter, drawn from any point within a circle, will cut the circumference.

THEO-

THEOREM V.

If, to the circumference of a circle (AFEB), *from any point* (D) *which is not the center, right-lines* (DA, DF, DE) *be drawn, the greateſt of all* (DA) *ſhall be that which paſſes through the center* (C); *and, of the reſt, that* (DF) *whoſe other extreme* (F) *is placed neareſt, in the circumference, to the extreme* (A) *of the greateſt, will exceed any other* (DE) *whoſe extreme* (E) *is at a greater diſtance.*



From the center C, let CE and CF be drawn.

1. AD (= DC + CF[r]) ⌐ DF[s]. [r] Ax. 4. [s] 19. 1.
2. Since DC is common, CF = CE, and DCF ⌐ DCE[t], therefore is DF ⌐ DE[u]. [t] Ax. 2. [u] 21. 1.

COROLLARY I.

Becauſe no two lines, DE, DF, drawn from D, on the ſame ſide of the diameter AB, can be equal to each other[w], three equal right-lines cannot poſſibly be drawn from the periphery to any point, beſides the center of the circle: and, therefore, if from a point in any circle, three equal right-lines can be drawn to the periphery, that point is the center of the circle. [w] 5. 3.

COROLLARY II.

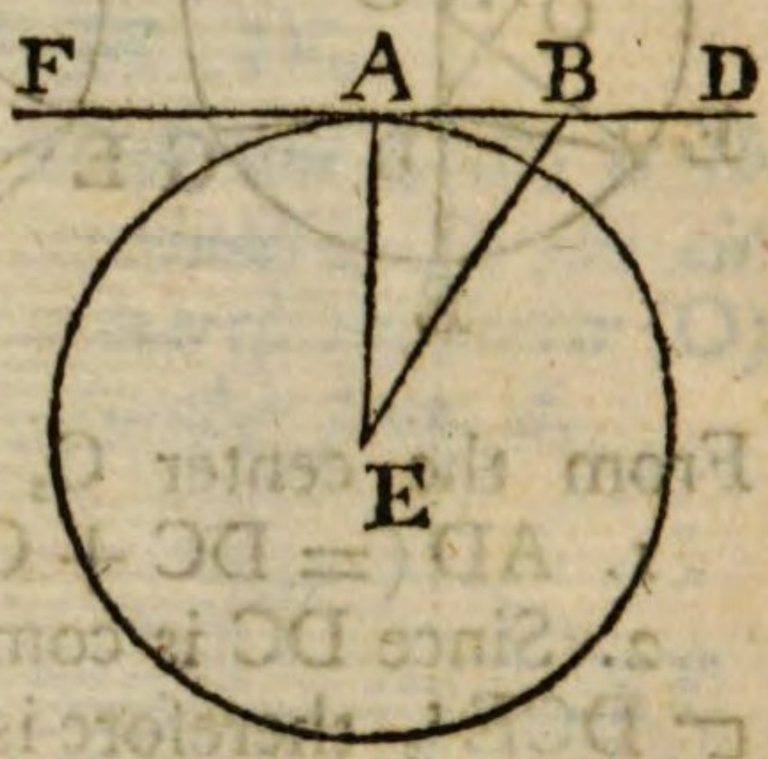
Hence it alſo follows, that no circle can be deſcribed to cut another FBG in more points than two: for, if it were poſſible to cut it in three points G, E, F, then right-lines drawn from the center

Q,

[x] Def. 33. 1. [y] Cor. 1. to 4. 3. Q, to thoſe points, would be all equal [x], which is ſhewn to be impoſſible [y], unleſs when the center Q coincides with C; and then the circles themſelves will neither cut, nor touch, but coincide, and become one circle [x].

THEOREM VI.

A right-line (FD) *drawn through any point* (A) *in the circumference of a circle, at right-angles to the radius* (EA) *terminating in that point, will touch the circle.*



From any point in FD, to the center E, let the right-line BE be drawn; which being greater than AE [a], the point B muſt, neceſſarily, fall out of the circle [b]: and therefore, as the ſame argument holds good with regard to every other point in the line FD (except A), it is manifeſt that this line cuts off no part of the circle, but touches it, in one point only.

[a] 20. 1.

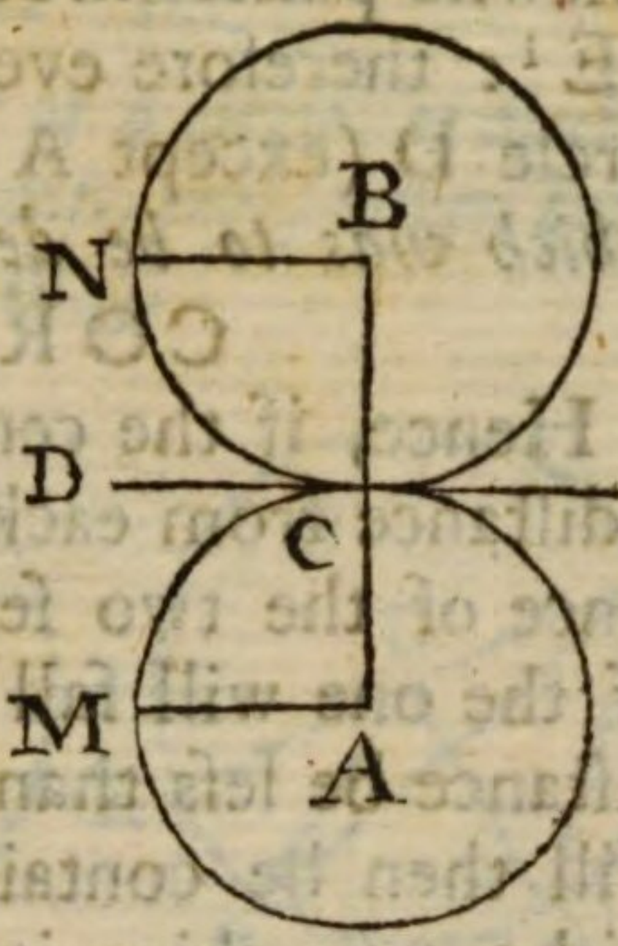
[b] Def. 33. and Ax. 2. of 1.

THEOREM VII.

If the diſtance (AB) *of the centers of two circles, be equal to the ſum of the two ſemi-diameters* (AM, BN), *the circles will touch each other, outwardly; and the right-line* (AB) *joining their centers, will paſs through the point of contact.*

In

In AB, take AC = AM, and let DCE be drawn perpendicular to AB: then, BC being alſo = BN [c], the circumferences of both circles will paſs through the point C [d]: but the right-line DE *(by the precedent)* falls wholly above the one, and wholly below the other; therefore the circles themſelves fall wholly without each other, and touch in one point C only.



[c] Conſtr. and Ax. 5.

[d] Def. 33. 1.

COROLLARY.

Hence, if the centers of two circles be placed at a diſtance, from one another, leſs than the ſum of the two ſemi-diameters, a part, at leaſt, of the one will be contained within the other: but, if the diſtance be greater than that ſum, the two circles will then neither touch, nor cut each other.

THEOREM VIII.

If the diſtance (CD) *of the centers of two circles* (CAF, CAE) *be equal to the difference of the two ſemi-diameters* (CA, DE), *then will thoſe circles touch inwardly; and that radius* (CA) *of the greater, which is drawn through the center* (D) *of the leſſer, will meet the two peripheries in the point of contact.*

From any point E in the circumference of the leſſer, to the two centers, let EC and ED be drawn. Becauſe CA exceeds DE by the line DC [e], or becauſe DE + DC = CA [f] = DA + DC [g], therefore is DA = DE [h]; and ſo the circumference of the circle D

[e] Hyp.

[f] Ax. 4.

[g] Ax. 3.

[h] Ax. 5.

likewiſe

likewiſe paſſes through A: but CA is greater than CE [i]: therefore every point in the periphery of the circle D (except A only) falls within the circle C: *which was to be demonſtrated.*

i 5. 3.

COROLLARY I.

Hence, if the centers of two circles be placed at a diſtance from each other, greater than the difference of the two ſemi-diameters, a part, at leaſt, of the one will fall without the other; but, if the diſtance be leſs than that difference, the leſſer circle will then be contained wholly in the greater, but without touching it.

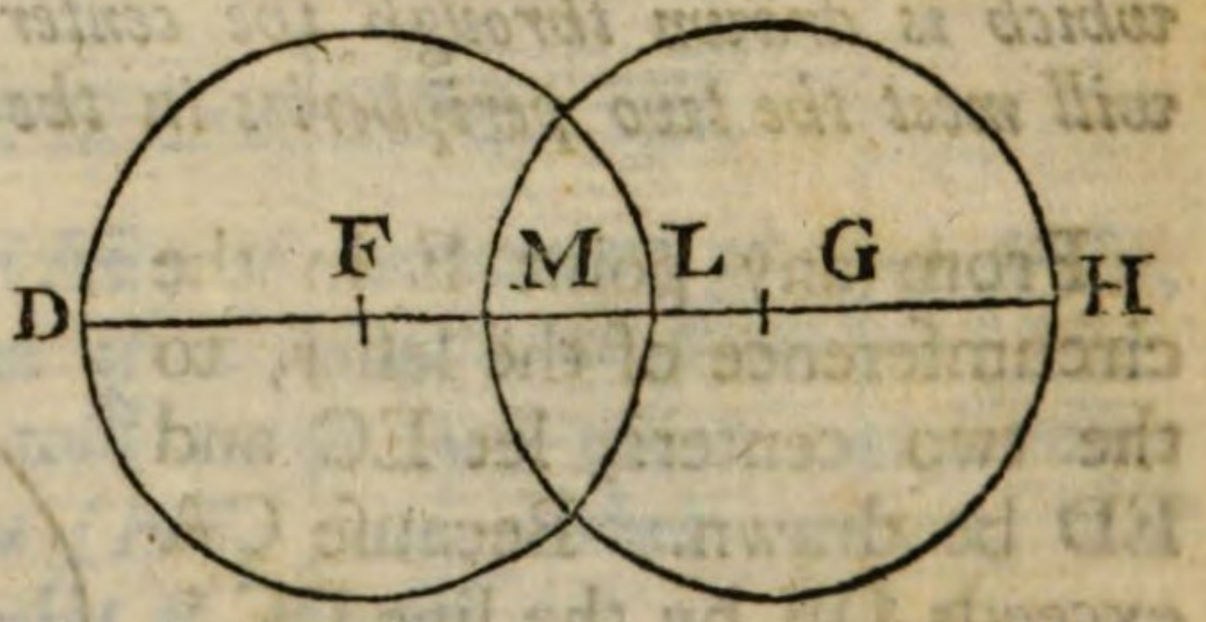
COROLLARY II.

Hence, and from *the precedent*, it likewiſe appears, that if two circles touch, either inwardly or outwardly, a right-line, drawn through their two centers, will alſo paſs through the point of contact: becauſe they can only touch, when the diſtance of their centers is equal to the ſum, or to the difference of their ſemi-diameters [k].

k Cor. of 7. and Cor. 1. of 8.

THEOREM IX.

If the diſtance of the centers (F, G) *of two circles* (DL, MH) *be leſs than the ſum, and greater than the difference of the two ſemi-diameters* (FL, GM), *thoſe circles will cut each other.*



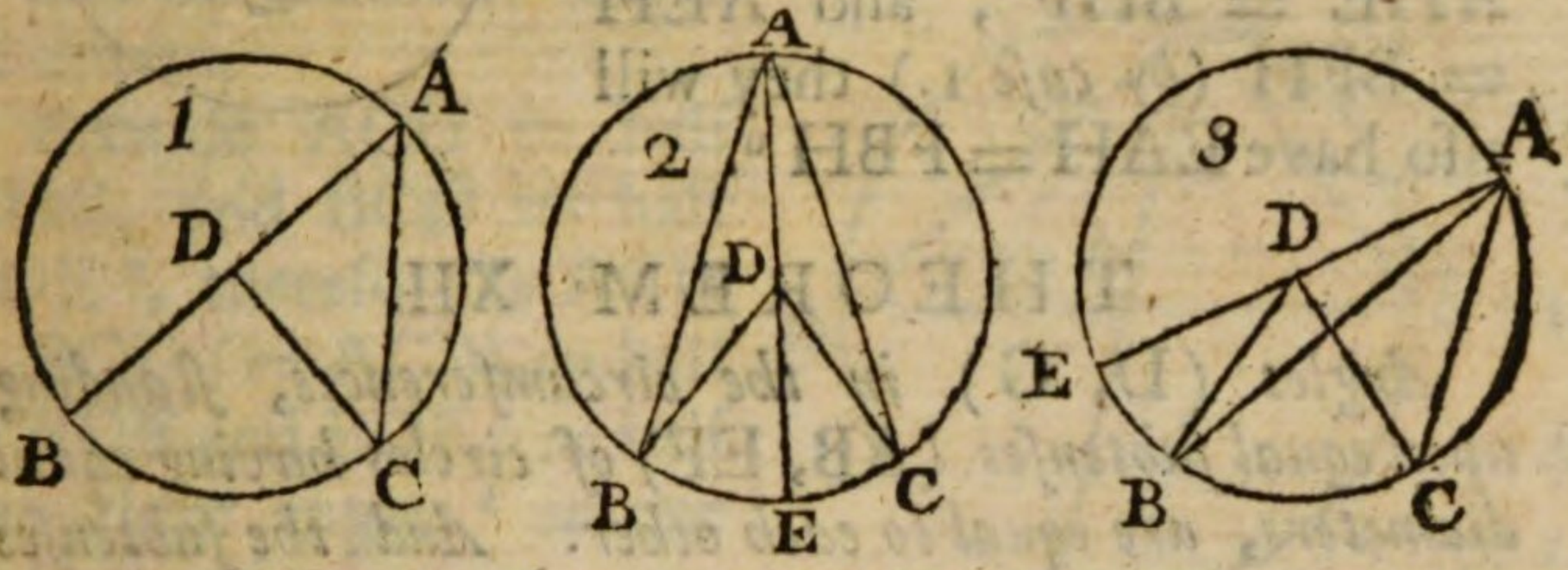
For, ſince the diſtance of the two centers is ſuppoſed leſs than the ſum of the ſemi-diameter, a part of the one circle MH, falls within the other DL [l]; and ſince that diſtance is greater than the difference of thoſe ſemi-diameters, a part of the ſame circle MH alſo falls without the circle DL [m]: *which was to be proved* [n].

l Cor. to 7. 3.
m Cor. to 8. 3.
n Def. 11. of 3.

THEO-

THEOREM X.

The angle (BDC) *at the center of a circle, is double to the angle* (BAC) *at the circumference, when both angles ſtand upon the ſame arch* (BC).



Let the diameter ADE be drawn.

In the firſt caſe (where AB paſſes through the center) BDC = A + C ° = 2A ᵖ. ° 9. 1.

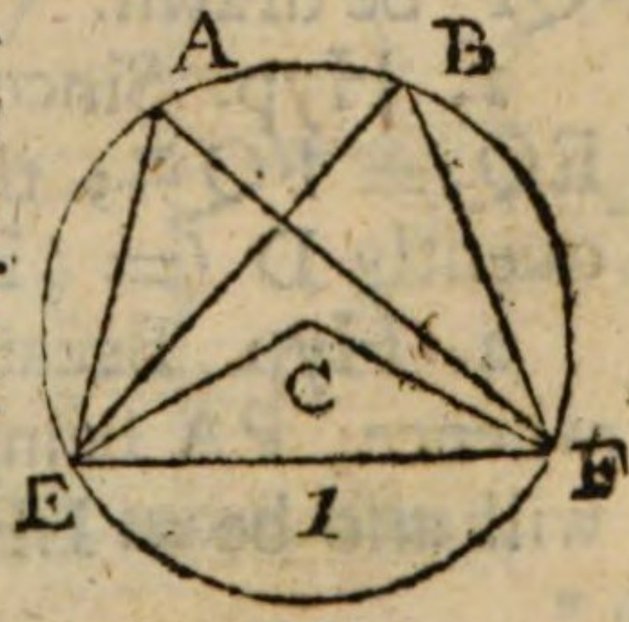
In the ſecond caſe, BDE = 2BAE (*by caſe* 1.); ᵖ 12. 1. to which adding CDE = 2CAE, we have BDC = 2BAC ᵠ. ᵠ Ax. 4.

In the third caſe, CDE = 2CAE (*by caſe* 1.) from whence ſubtracting BDE = 2BAE, there remains BDC = 2BAC ʳ. ʳ Ax. 5.

THEOREM XI.

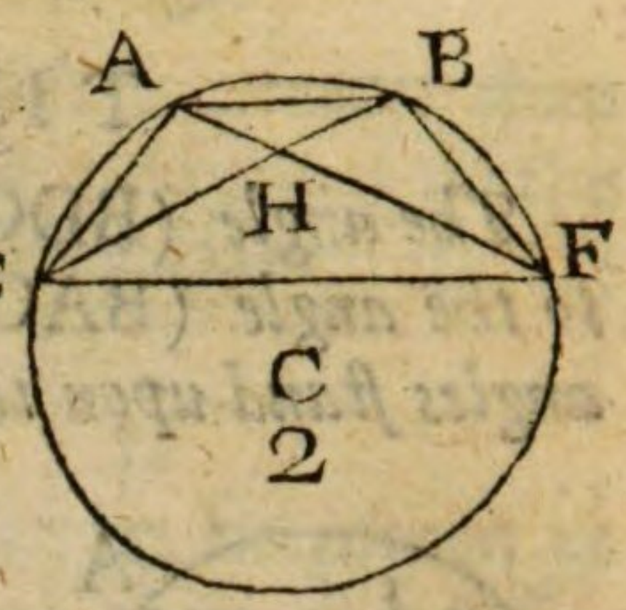
All angles (EAF, EBF) *in the ſame ſegment* (EABF) *of a circle, are equal to each other.*

Case I. *If the ſegment be greater than a ſemi-circle*; from the center C draw CE and CF; then EAF and EBF being each of them = to half ECF ˢ, they muſt neceſſarily be equal to each other.



ˢ 16. 3.

CASE II. *If the ſegment be leſs than a ſemi-circle*; let H be the interſection of EB and AF: then the triangles AEH and BFH, having the angle AHE = BHF [z], and AEH = BFH (*by caſe* 1.) they will alſo have EAH = FBH [a].

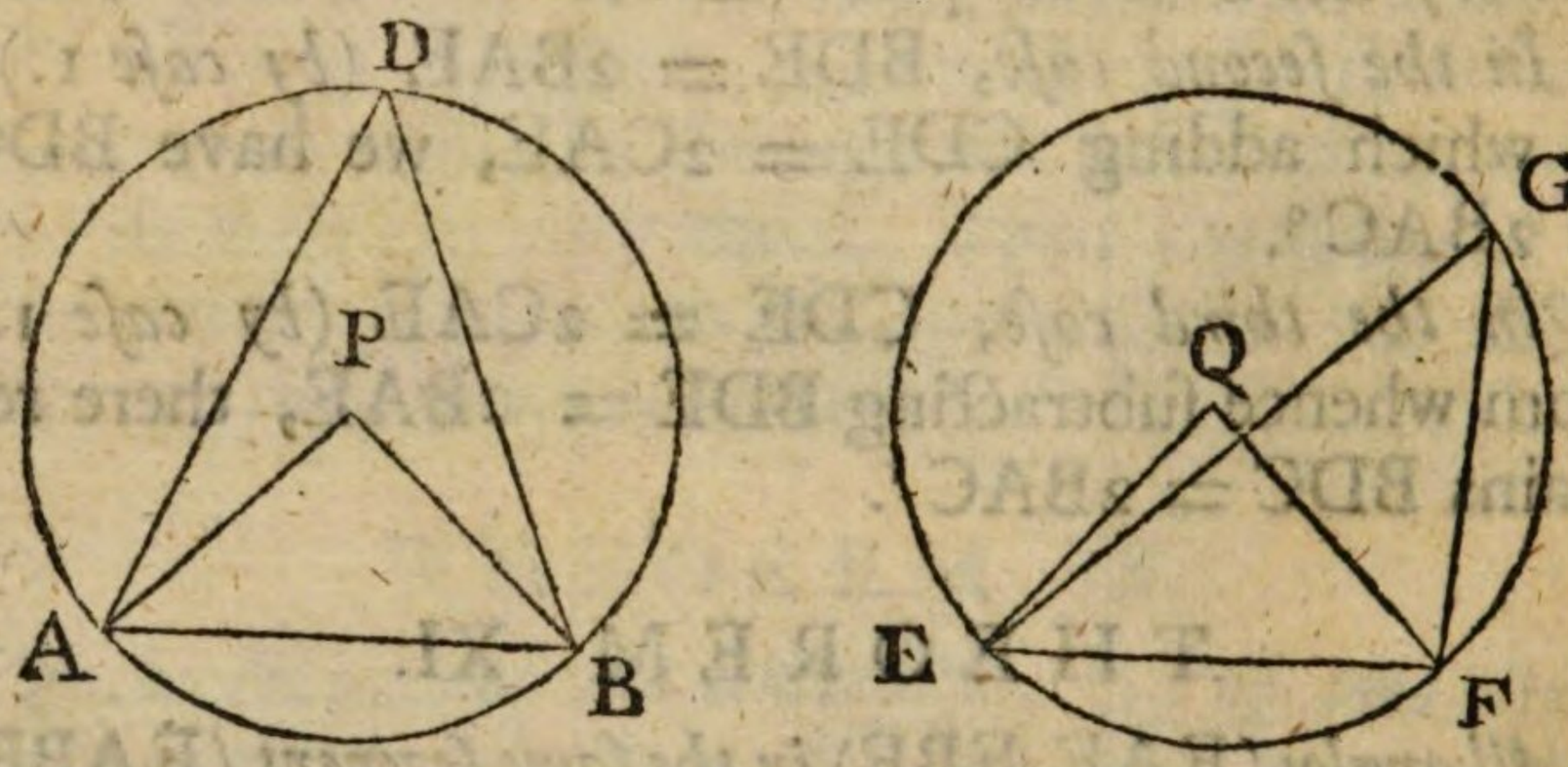


[z] 3. 1.

[a] Cor. 1. to 10. 1.

THEOREM XII.

Angles (D, G) *in the circumferences, ſtanding upon equal ſubtenſes* (AB, EF) *of circles having equal diameters, are equal to each other. And the ſubtenſes of equal angles, in the circumferences of circles having equal diameters, are alſo equal.*



From the centers P, and Q, let PA, PB, QE, QF be drawn.

1. Hyp. Since AB = EF [w], and AP = BP [x] = EQ = FQ [w]; therefore is P = Q [y], and conſequently D (= ½P [z] = ½Q) = G.

2. Hyp. Becauſe D = G, therefore P = Q [z]; whence, PA being = QE, and PB = QF [w], AB will alſo be = EF [a].

[w] Hyp.

[x] Def. 33. of 1.

[y] 14. 1.

[z] 10. 3.

[a] Ax. 10. of 1.

COROLLARY.

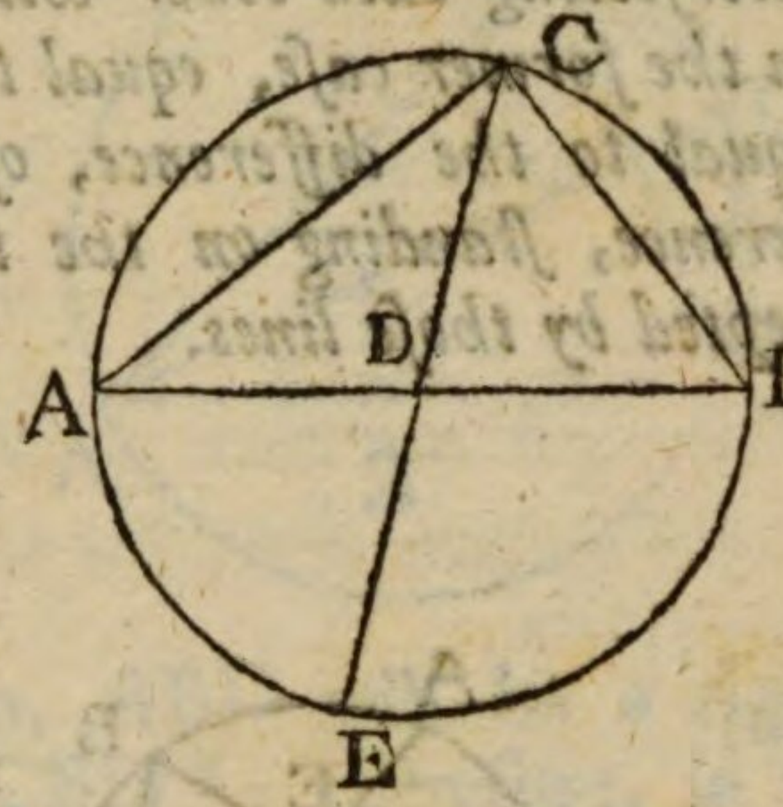
Hence angles in the circumference, ſtanding upon equal chords of the ſame circle, are equal.

THEOREM

THEOREM XIII.

The angle (ACB) *in a ſemi-circle, is a right angle.*

Let the diameter CDE be drawn.



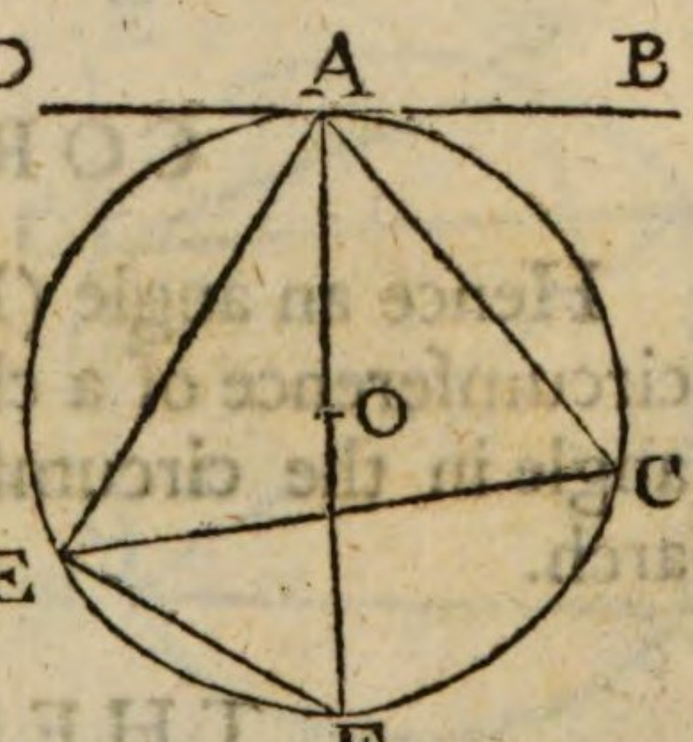
Becauſe ACD = half ADE, and BCD = half BDE [b], therefore is ACD + BCD (= ACB) = half of ADE and BDE [c] = half two right angles [d] = one right angle.

[b] 10. 3.
[c] Ax. 4. 1.
[d] 1. 1.

THEOREM XIV.

The angle (CAB) *included by a tangent to a circle and a chord* (AC) *drawn from the point of contact* (A), *is equal to the angle* (AEC) *in the alternate ſegment.*

Let the diameter AOF be drawn, and E, F be joined.

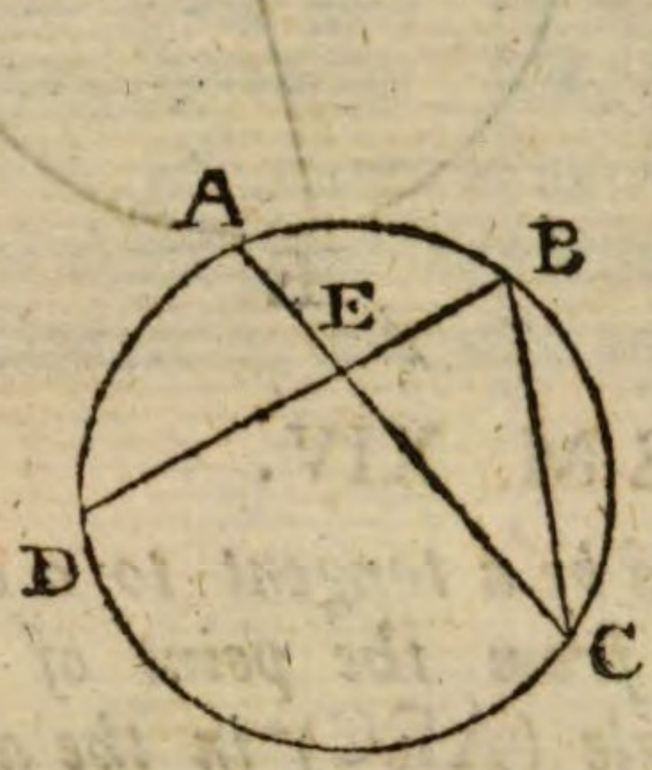


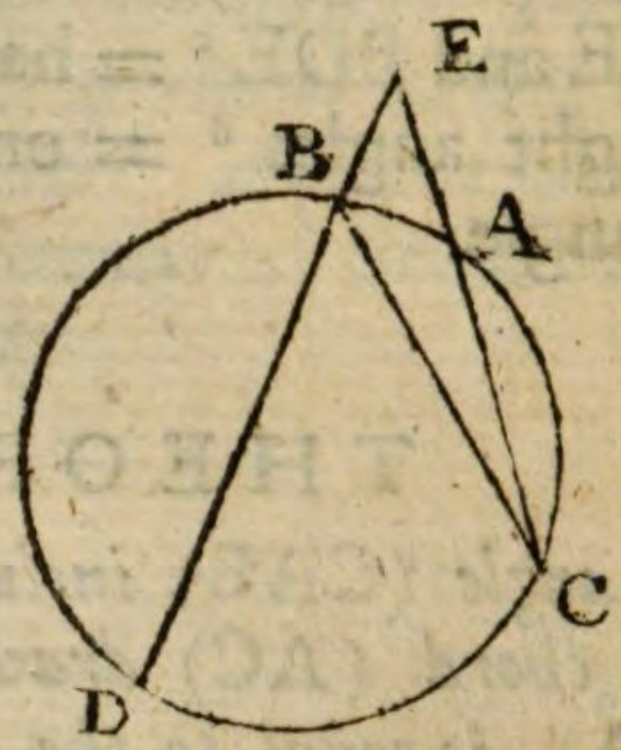
The line DB falling wholly above the circle [e], OA is the leaſt line that can be drawn to it from the center O [f]; and OAB is therefore a right-angle [g]: but FEA is alſo a right-angle [h]: therefore, if from theſe equal angles, the equal [i] angles FAC, FEC (ſtanding on the ſame arch FC) be taken away, there will remain BAC = AEC [k].

[e] Def. 9. 3.
[f] Def. 33. 1. and Ax. 2.
[g] 20. 1.
[h] 13. 3
[i] 11. 3.
[k] Ax. 5.

THEOREM XV.

The angle (DEC) *made by two lines* (DEB, CEA) *interſecting each other within, or without a circle, is, in the former caſe, equal to the ſum, and in the latter, equal to the difference, of two angles in the circumference, ſtanding on the two arcs* (DC, AB) *intercepted by thoſe lines.*





Let the chord CB be drawn.

[e] 9. 1. Then DEC = DBC + ACB[e], *in the firſt caſe.*

[f] 9. 1. and Ax. 5. And DEC = DBC − ACB[f], *in the ſecond caſe.*

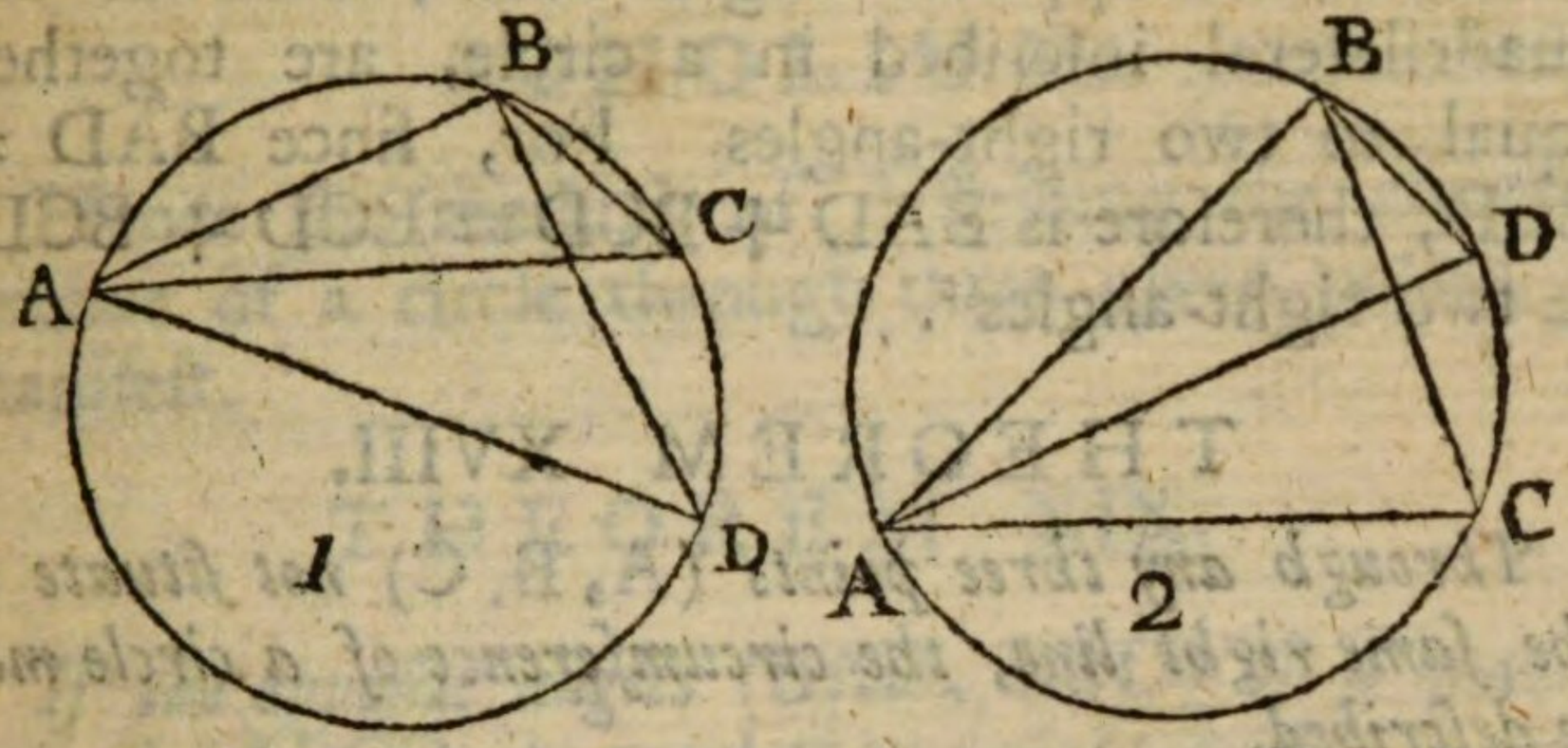
COROLLARY.

Hence an angle (E) formed below, or above the circumference of a circle, is greater, or leſs than an angle in the circumference, ſtanding on the ſame arch.

THEOREM XVI.

The vertical angle (ABC) *of any oblique-angled triangle* (ACB) *inſcribed in a circle* (ABCD) *is greater, or leſs than a right-angle, by the angle* (CAD) *comprehended*

prehended under the baſe (AC) *and the diameter* (AD) *drawn from the extremity of the baſe.*



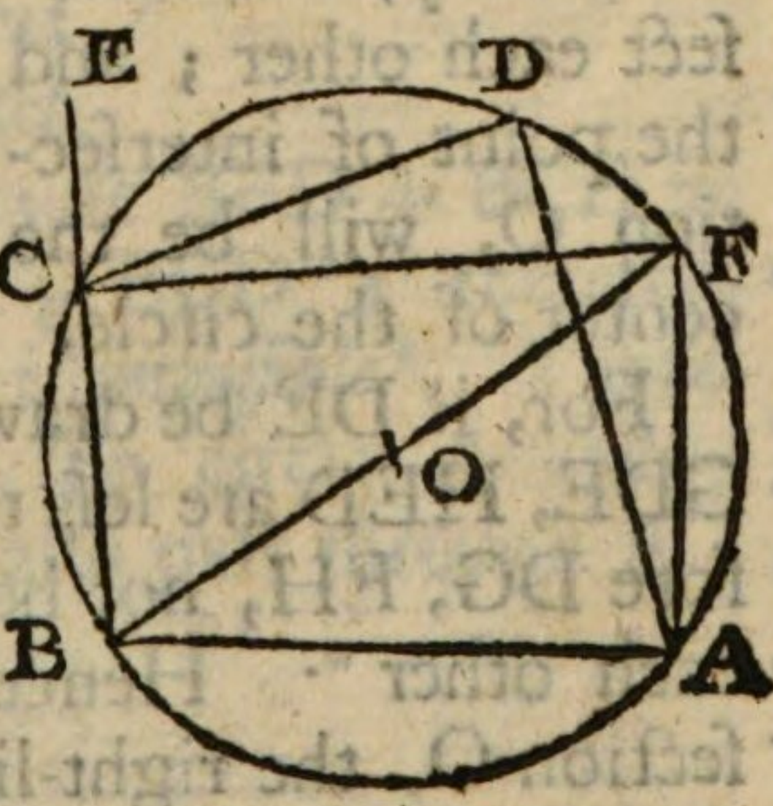
For, BD being drawn, ABD will be a right-angle [i], and CAD = CBD [k]; therefore, in the firſt caſe, ABC = right-angle + CAD [l]; and in the ſecond, ABC = right-angle — CAD [m].

[i] 13. 3.
[k] 11. 3.
[l] Ax. 4.
[m] Ax. 5.

THEOREM XVII.

If any ſide (BC) *of a quadrilateral* (ABCD) *inſcribed in a circle, be produced out of the circle, the external angle* (ECD) *will be equal to the oppoſite, internal angle* (BAD).

Let the diameter BF be drawn, and let AF and CF be joined: then the angle BAF being a right angle (= BCF) = ECF [n], and DAF alſo = DCF [o], ſtanding both on the ſame arch DF; thence will the remainders BAD and ECD be alſo equal [q].



[n] 13. 3.
[o] 11. 3.
[q] Ax. 5.

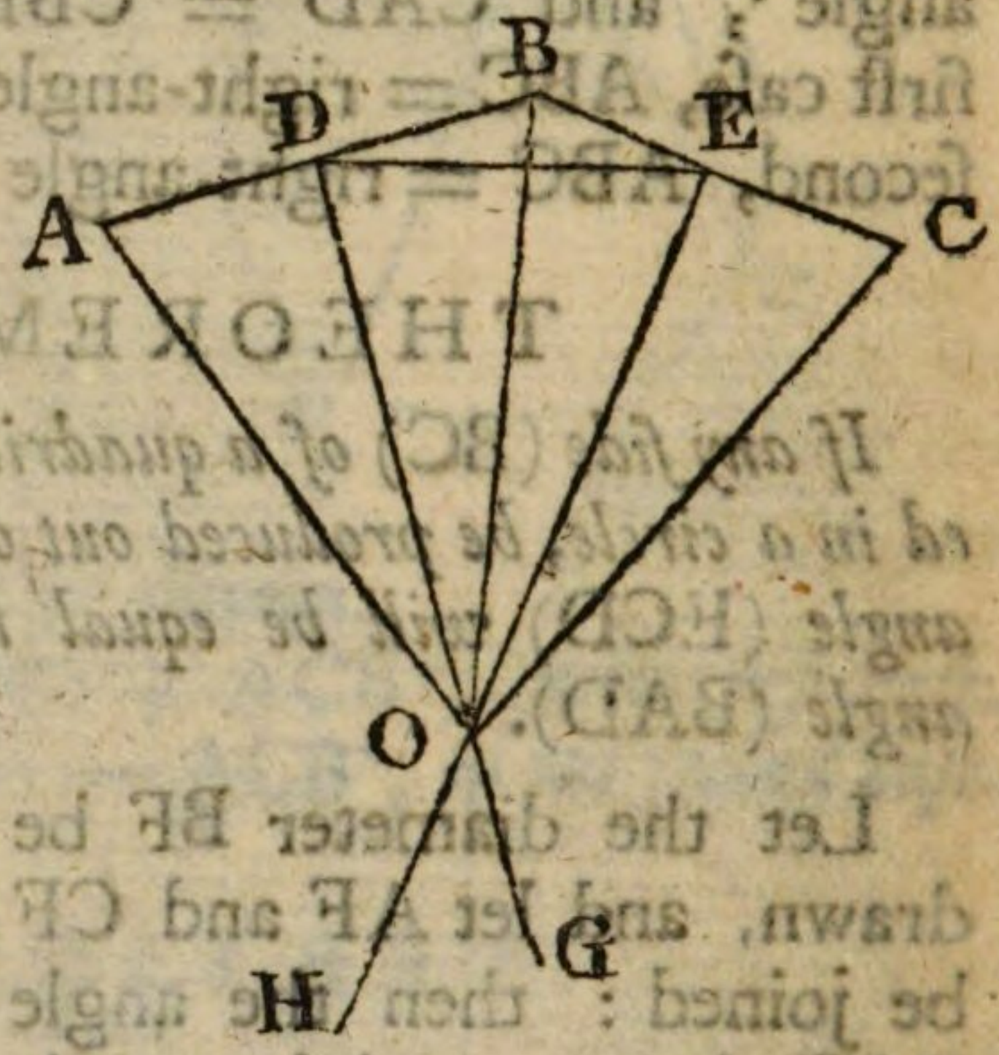
COROLLARY.

Hence the oppoſite angles BAD, BCD of any quadrilateral inſcribed in a circle, are together, equal to two right-angles. For, ſince BAD = ECD, therefore is BAD + BCD = ECD + BCD = two right-angles [s].

[r] Ax. 4.
[s] 1. 1.

THEOREM XVIII.

Through any three points (A, B, C) *not ſituate in the ſame right line, the circumference of a circle may be deſcribed.*



Draw AB and BC, which let be biſected by the perpendiculars DG and EH, infinitely produced on that ſide of AB or BC, on which the angle ABC is formed.

Theſe perpendiculars, I ſay, will interſect each other; and the point of interſection O, will be the center of the circle.

For, if DE be drawn, it is plain, that the angles GDE, HED are leſs than two right-angles [t]; therefore DG, EH, not being parallels [u], they will meet each other [w]. Hence, if from the point of interſection O, the right-lines OA, OB, OC be drawn, the triangles ADO, BDO, having two ſides equal, each to each [x], and the angles ADO, BDO, contained by them, equal [y], will likewiſe have AO = BO [z]. After the very ſame manner is CO = BO; therefore AO = BO = CO [a]; whence the circumference of a circle deſcribed from the center O, at the

[t] Ax. 2.
[u] Cor. 2. to 7. 1.
[w] Def. 13. of 1.
[x] Conſtr.
[y] Ax. 7.
[z] Ax. 10.
[a] Ax. 1.

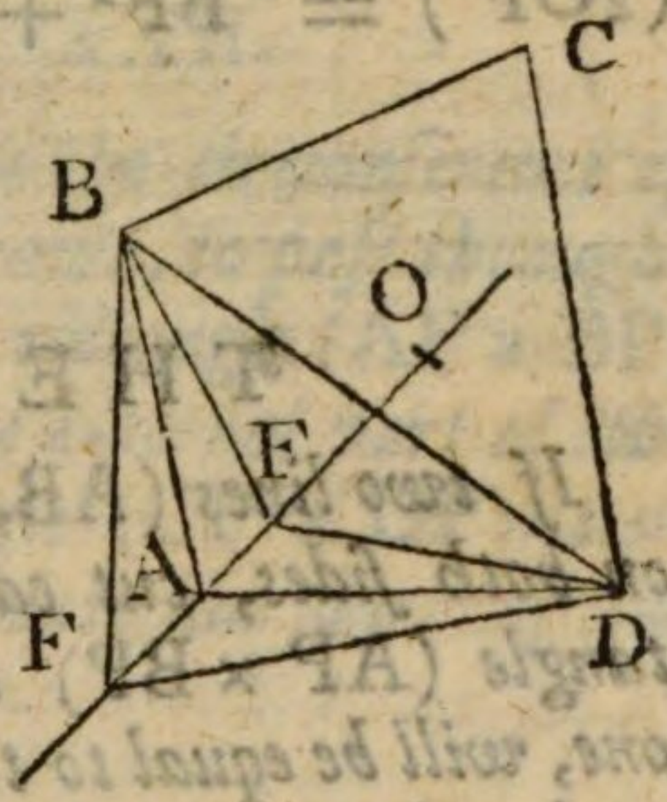
the distance of AO, will also pass through B and C [b]. [b] Def. 33. 1.

SCHOLIUM.

Hence the method of describing the circumference of a circle through three given points, is manifest.

THEOREM XIX.

If the opposite angles (BAD, BCD) *of a quadrilateral* (ABCD) *be equal to two right-angles, a circle may be described about that quadrilateral.*

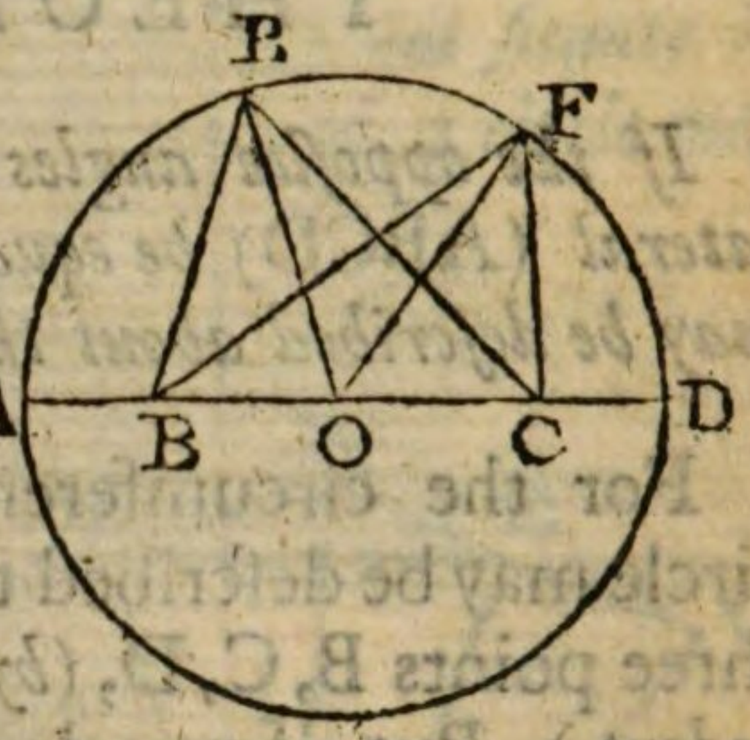


For the circumference of a circle may be described thro' any three points B, C, D, (*by the precedent*.) But, if you deny that it passes thro' A; then, thro' the center O, let OAF be drawn, and let it (if possible) pass through some other point F in the line OAF, (for it must cut this line somewhere [c]); also let BF and DF be drawn. Because BFD + BCD = two right-angles [d] = BAD + BCD [e]; therefore must BFD = BAD [f]: *which is impossible* [g]. Therefore the circumference of the circle described through B, C and D, must also pass through A.

[c] Cor. to 4. 3.
[d] Cor. to 17. 3.
[e] Hyp.
[f] Ax. 5.
[g] 23. 1.

THEOREM XX.

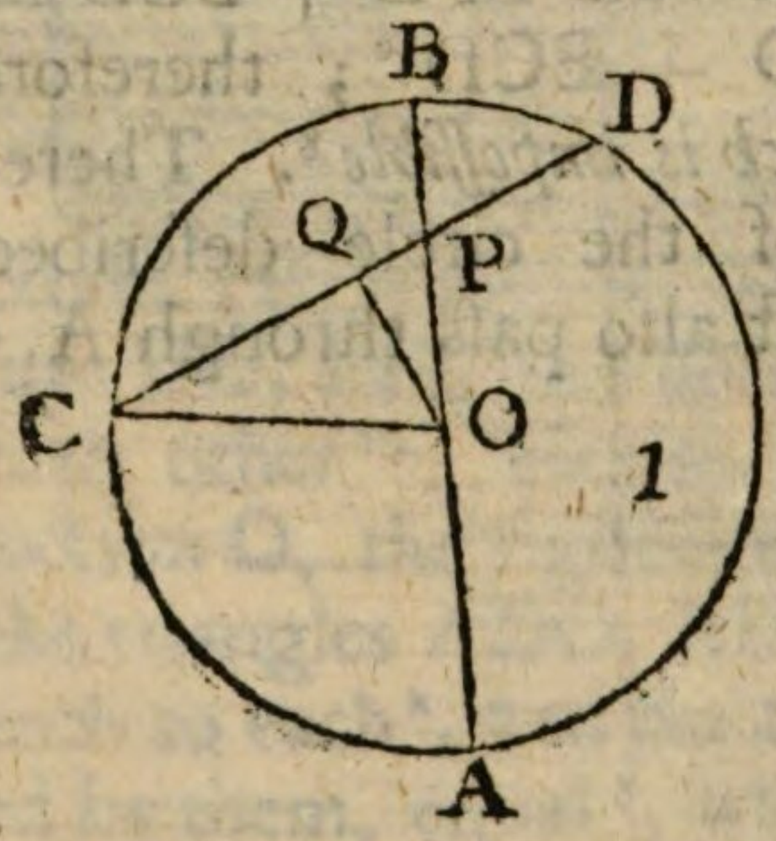
If from two points (B, C) *in the ſame diameter* (AD), *equally diſtant from the center* (O) *of a circle, right-lines* (BE, CE; BF, CF) *be drawn to meet, two by two, in the circumference; the ſum of the ſquares of any two correſponding ones, will be equal to the ſum of the ſquares of any other two, meeting in like manner.*

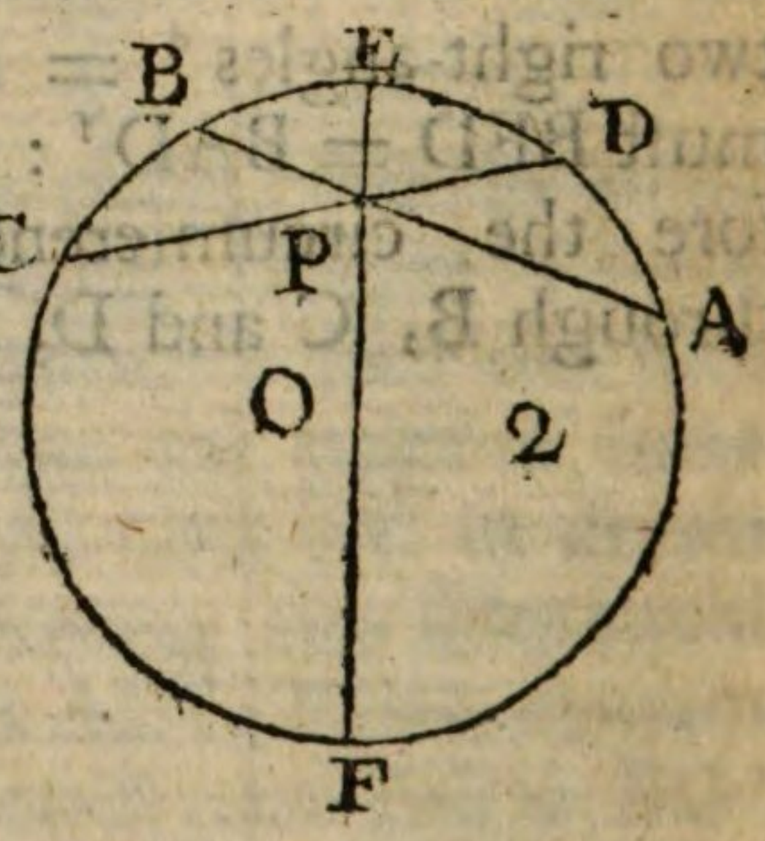


For, if OE and OF be drawn; then will $BE^2 + CE^2 = {}^{h}\ 2BO^2 + 2OE^2$ (11. 2.) $(2OF^2) = {}^{h}\ BF^2 + CF^2$.

THEOREM XXI.

If two lines (AB, CD), *terminated by the periphery on both ſides, cut each other within a circle, the rectangle* (AP × BP) *contained under the parts of the one, will be equal to the rectangle* (CP × DP) *contained under the parts of the other.*





Case I. If one of the two lines (AB) paſſes through the center O; then let OQ be drawn perpendicular to the other CD, and let OC be joined.

It

It is plain, becauſe QD = QC [i], that DP is equal to the difference of the ſegments CQ and PQ [k]: but the rectangle under the ſum and difference of the two ſides OC, OP, of any triangle COP, is equal to the rectangle under the whole baſe CP, and the difference of its two ſegments [l]; therefore, the ſum of the two ſides OC, OP being (= OA + OP) = AP, and their difference (= OB — OP) = BP, thence is the rectangle contained under AP and BP equal to the rectangle contained under CP and DP.

[i] 2. 3.

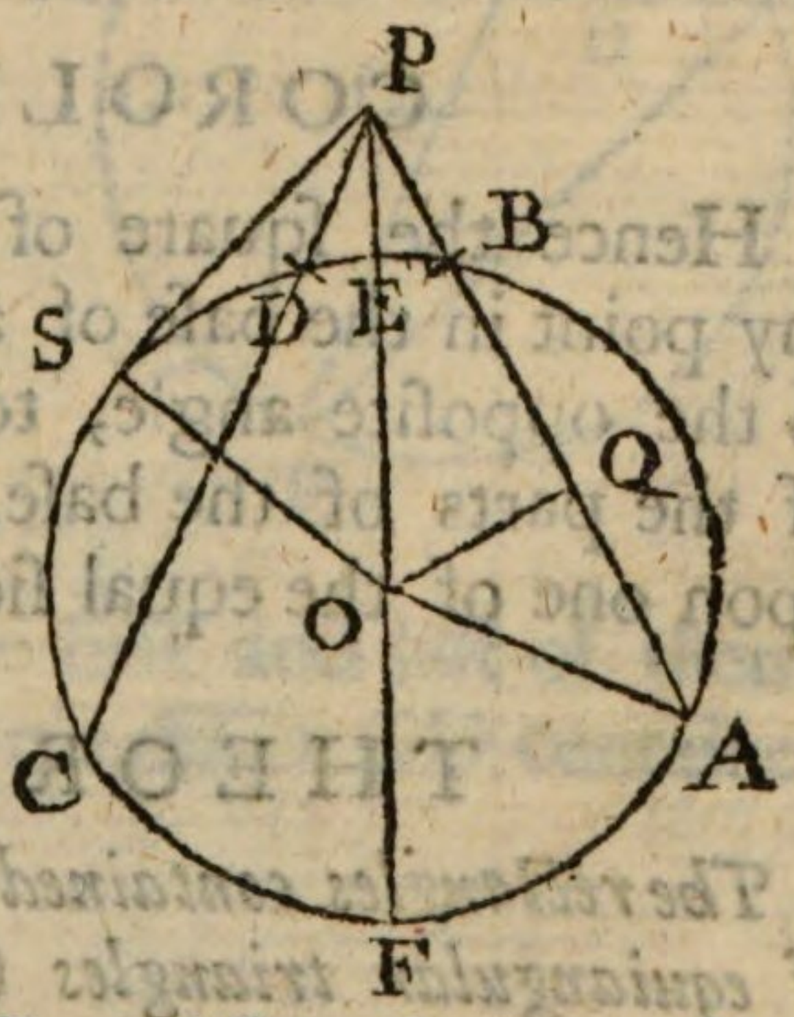
[k] Ax. 5.

[l] Cor. 1. to 5. 2.

CASE II. If neither of the two lines paſs through the center: let the diameter EPF be drawn; then, *by the preceding caſe*, AP × BP = FP × EP = CP × DP.

THEOREM XXII.

If from two points (A, C) *in the circumference of a circle, two lines* (AP, CP) *be drawn, to paſs through, and meet without the circle; the rectangle* (AP × BP) *contained under the whole and the external part of the one, will be equal to the rectangle* (CP × DP) *contained under the whole and the external part of the other.*



Through the center O, let PF be drawn, meeting the circumference in E and F; let OQ be perpendicular to AP, and let A, O be joined.

Then, (*by Cor.* 1. *to* 9. 2.) the rectangle contained under PF (= PO + OA) and PE (= PO — OA) is = the rectangle contained under AP and PB. After the very ſame manner, PF × PE = CP × DP: therefore AP × BP = CP × DP [m].

[m] Ax. 1.

COROL.

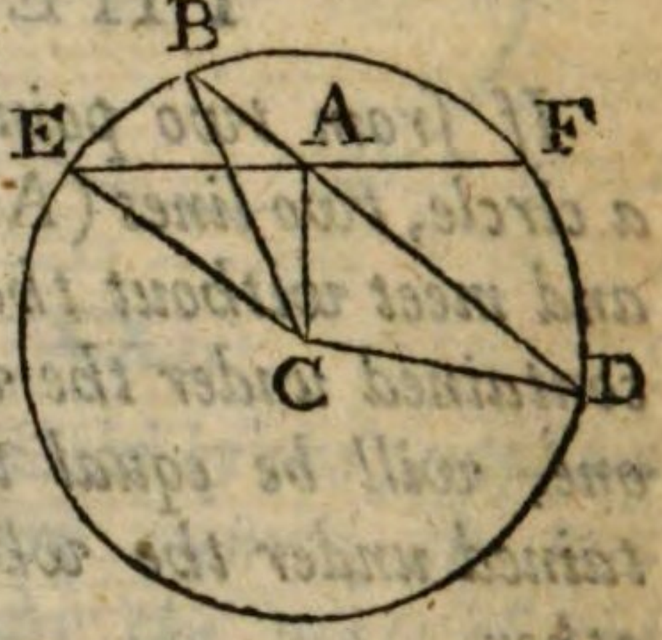
COROLLARY I.

Hence, if PS be a tangent at S, and the radius OS be drawn; then, PF being = the ſum of PO and OS, and PE = their difference; it follows, that $PS^2 = PF \times PE$ [o] $= PC \times PD$.

[o] Cor. to 8. 2.

THEOREM XXIII.

If from the center (C) *of a circle, to a point* (A) *in any chord* (BD), *a line* (CA) *be drawn; the ſquare of that line, together with the rectangle contained under the two parts of the chord, will be equal to a ſquare made upon the radius of the circle.*



Let EAF be another chord, perpendicular to CA, and let C, E be joined.

[q] 2. 3. [r] 1. 2. [s] 21. 3. Since AF = AE [q], thence will $AE^2 = AE \times AF$ [r] = $AB \times AD$ [s]; to which equal quantities adding AC^2, we have [t] $CE^2 = AB \times AD + AC^2$.

[t] 8. 2.

COROLLARY II.

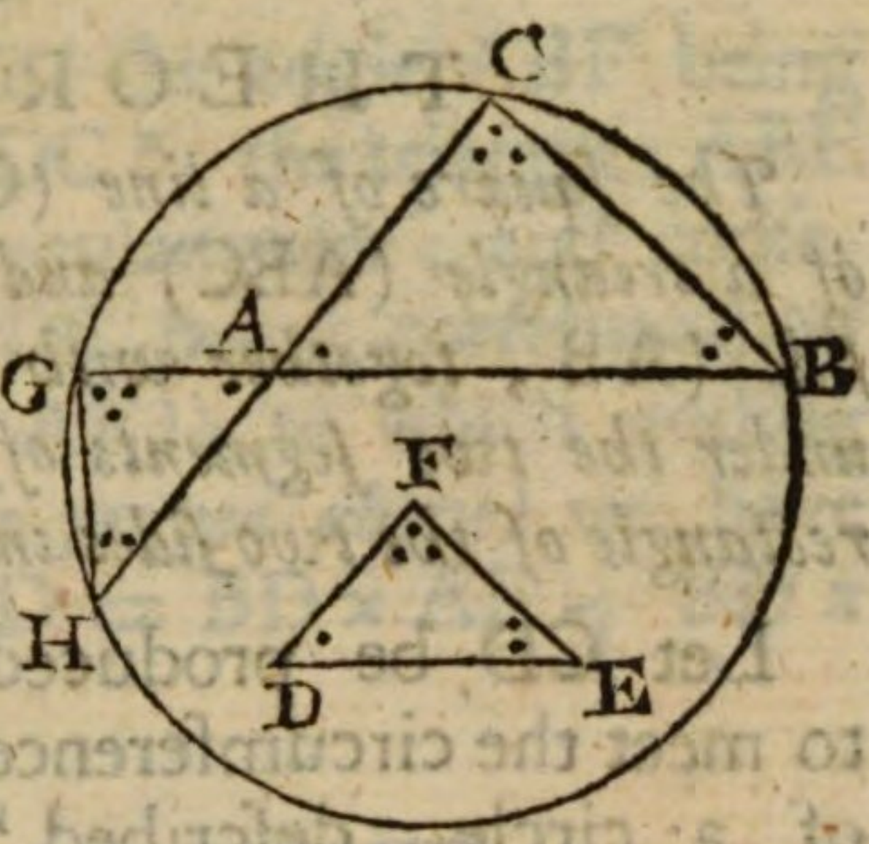
Hence the ſquare of a line (AC) drawn from any point in the baſe of an iſoſceles triangle (BCD) to the oppoſite angle, together with the rectangle of the parts of the baſe, is equal to a ſquare made upon one of the equal ſides of the triangle.

THEOREM XXIV.

The rectangles contained under the correſponding ſides of equiangular triangles (ABC, DEF) *taken alternately, are equal.*

I ſay, if A = D, B = E and C = F, then will $AB \times DF = AC \times DE$.

In



In BA produced, let AG be taken = DF; let GCB be the circumference of a circle passing through the three points B, C, G [w], meeting CA produced in H; and let G, H be joined.

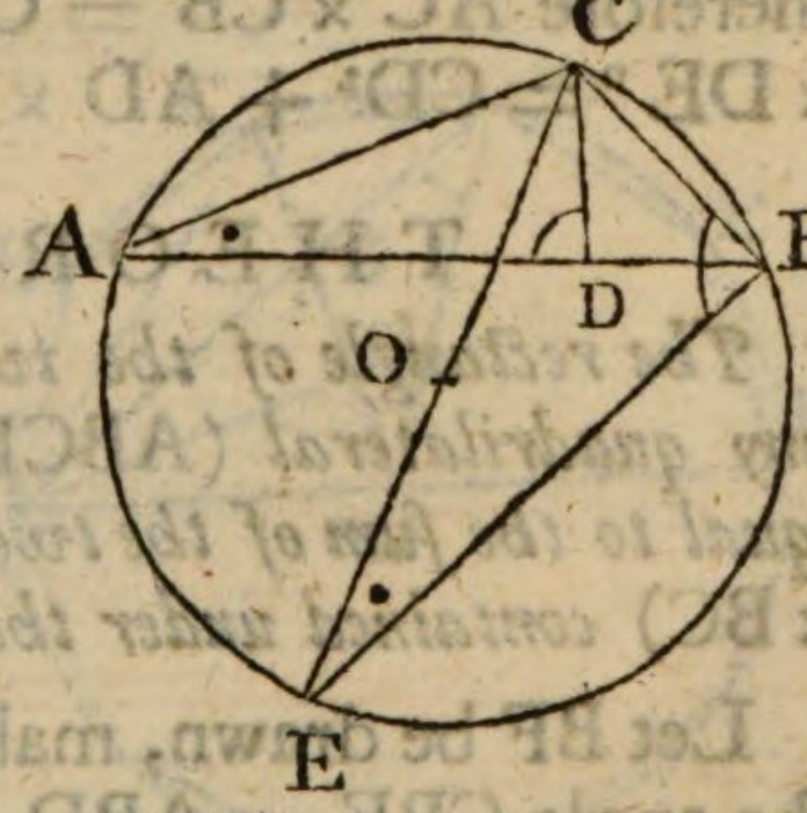
[w] 18. 3.

Because the angle H = B [y] = E [z], HAG = BAC = D [z], and AG = DF [z], thence is AH = DE [b]; and therefore AC × DE = AC × AH [c], = AB × AG [d] = AB × DF [c].

[y] 11. 3.
[z] Hyp.
[a] 3. 1.
[b] 15. 1.
[c] 1. 2.
[d] 21. 3.

THEOREM XXV.

The rectangle under the two sides (AC, BC) *of any triangle* (ABC), *is equal to the rectangle under the perpendicular* (CD) *to the base thereof, and the diameter* (CE) *of the circumscribing circle.*



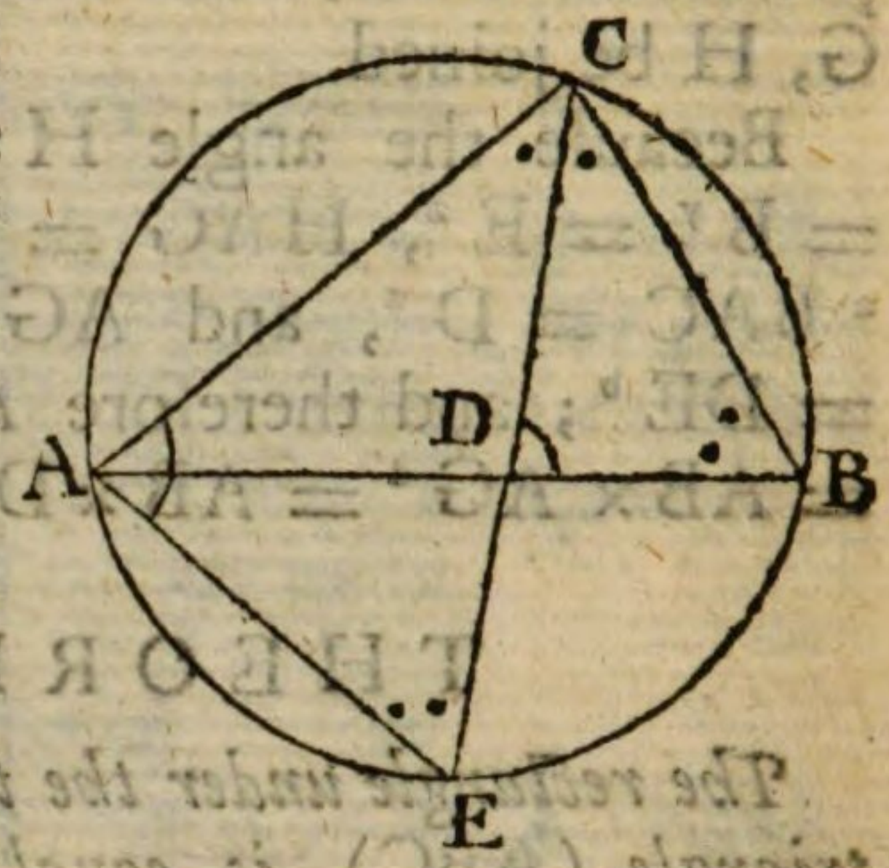
For, B, E being joined, the angles A, E will be equal [f], and ADC, EBC both right-angles [g]; and, consequently, the triangles ACD, ECB equiangular [h]: therefore AC, EC; CD, CB being corresponding sides, opposed to equal angles, the rectangle AC × CB, contained under the first and last of them, will be equal to the rectangle EC × CD contained under the other two [i].

[f] 11. 3.
[g] Hyp. and 13. 3.
[h] Cor. 1. to 10. 1.
[i] 24. 3.

THEO-

THEOREM XXVI.

The ſquare of a line (CD) *biſecting any angle* (C) *of a triangle* (ABC) *and terminating in the oppoſite ſide* (AB), *together with the rectangle* (AD × BD) *under the two ſegments of that ſide, is equal to the rectangle of the two ſides including the propoſed angle.*



Let CD be produced to meet the circumference of a circle, deſcribed [k] through the points A, C, B, in E; and let AE be drawn.

[k] 81. 3.

The angles E and B, ſtanding upon the ſame ſegment AC, are equal [l]; and ACE is equal DCB (by hypotheſis); therefore the triangles AEC, DCB are equiangular [m]; whereof AC, CD; CE, CB are correſponding ſides, oppoſed to equal angles: therefore AC × CB = CD × CE [n] = CD² + CD × DE [o] = CD² + AD × DB [p].

[l] 11. 3.

[m] Cor. 1. to 10. 1.

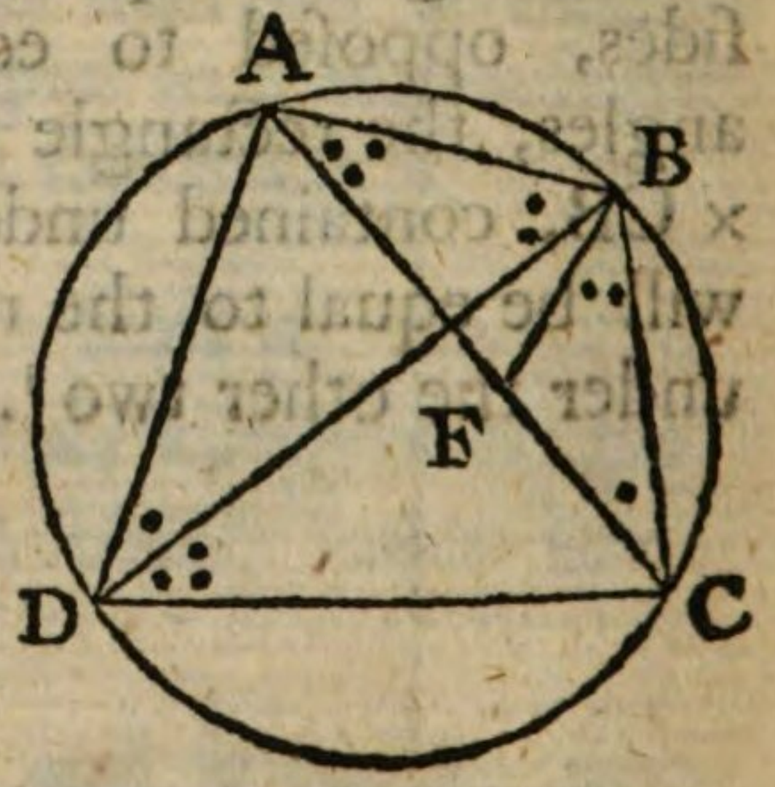
[n] 24. 3.

[o] 5. 2.

[p] 21. 3. and Ax. 4.

THEOREM XXVII.

The rectangle of the two diagonals (AC, BD) *of any quadrilateral* (ABCD) *inſcribed in a circle, is equal to the ſum of the two rectangles* (AB × DC, AD × BC) *contained under the oppoſite ſides.*



Let BF be drawn, making the angle CBF = ABD, and meeting AC in F.

Becauſe the angle BCF = ADB [q], and CBF = ABD [r], the triangles CBF, DBA are equiangular [s]; and therefore, BC, BD; CF, AD, being correſponding ſides, the rectangles BC × AD, and BD ×

[q] 21. 3.

[r] Conſtr.

[s] Cor. 1. to 10. 1.

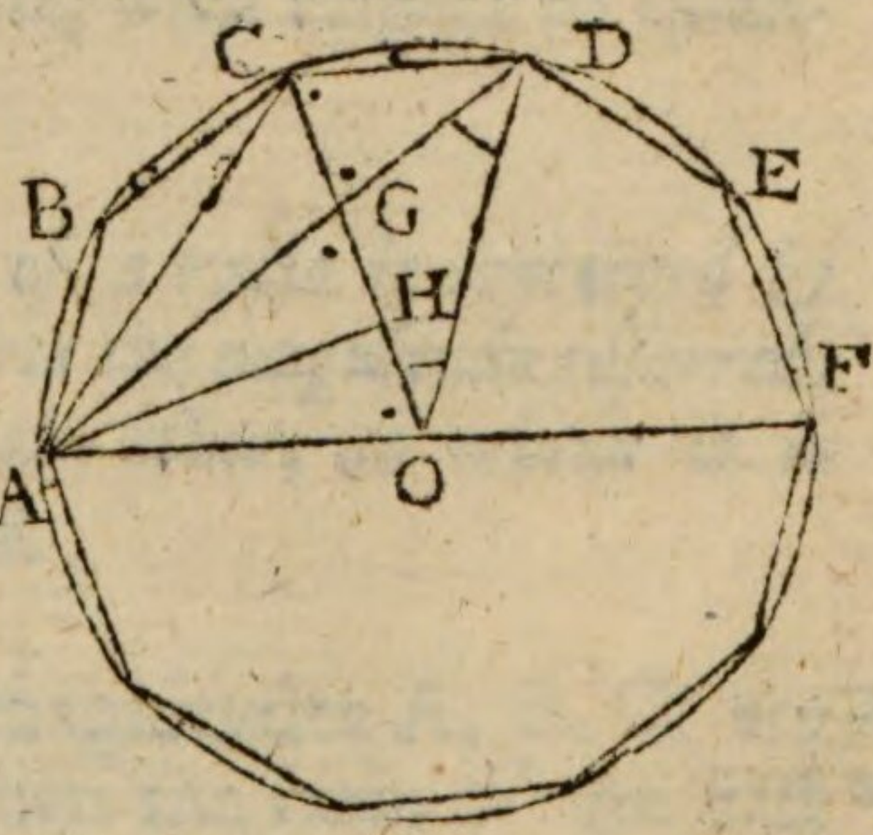
CF

CF will be equal [t]. Again, the angle ABF being = CBD [u], and BAF = BDC [w], the triangles ABF and BDC are, likewiſe, equiangular; and conſequently, AB, BD; AF, DC being correſponding ſides, AB × DC = BD × AF [t]; to which adding BC × AD = BD × CF *(ſo proved above)* we have alſo AB × DC + BC × AD = BD × AF + BD × CF = BD × AC [x].

[t] 24. 3.
[u] Ax. 4. 1.
[w] 11. 3.
[x] 5. 2.

THEOREM XXVIII.

If the radius of a circle (OADF) *be ſo divided into two parts, that the rectangle under the whole and the one part ſhall be equal to the ſquare of the other part; then will this laſt part be equal to the ſide* (CD) *of a regular decagon* (ABCDEF, &c.) *inſcribed in the circle; and that line whoſe ſquare is equal to the two ſquares of the whole, and of the ſame part, will be equal to the ſide* (AC) *of a regular pentagon inſcribed in the ſame circle.*



Draw the radii OA, OC, OD, OF; alſo draw AD, cutting OC in G, and let AH be perpendicular to OG.

The triangle ODG, having the angle COD (= ½DOF [y] = OAD [z]) = ODA [a], is iſoſceles [b]: moreover the triangle AOG, having AGO (= GDO + DOG [c] = 2DOC [a]) = AOC, is likewiſe iſoſceles [d]; as is alſo the triangle CDG, becauſe, CGD being = AGO [e], and CDG (CDA) = FAD [f], the triangles AOG, CDG are equiangular. Therefore, CD, AO; CG, GO being correſponding ſides, we have CG × AO (CG × CO) = CD × GO [g] = GO², becauſe GO = GD

[y] 1. 3.
[z] 10. 3.
[a] 12. 1.
[b] 18. 1.
[c] 9. 1.
[d] 18. 1.
[e] 3. 1.
[f] Cor. to 12. 3.
[g] 24. 3.

[h] 18. 1. $= GD = DC$ [h]: whence the former part of the propoſition is manifeſt.

[i] 16. 1. Again, becauſe $AG = AO$, HG will be $= HO$ [i]; and ſo GC being the difference of the ſegments HO and HC, we have (*by Cor.* 1. *to* 9. 2.) $AC^2 - AO^2 = CO \times CG = OG^2$ (*as above*); and conſequently $AC^2 = AO^2 + OG^2$.

The End of the THIRD BOOK.

ELEMENTS

OF

GEOMETRY.

BOOK IV.

DEFINITIONS.

1. RATIO is the proportion which one magnitude bears to another magnitude of the ſame kind, with reſpect to quantity.

The meaſure, or quantity of a ratio is conceived by conſidering what part, or parts the magnitude referred, called the antecedent, is of the other, to which it is referred, called the conſequent.

2. Three quantities, or magnitudes A, B, C, are ſaid to be proportional, when the ratio of the firſt A to the ſecond B, is the ſame as the ratio of the ſecond B, to the third C. A, B, C. 2. 4. 8.

3. Four quantities A, B, C, D, are ſaid to be proportional, when the ratio of the firſt A to the ſecond B, is the ſame as the ratio of the third C to the fourth D. A,B,C,D. 2.4.5.10.

To

To denote that four quantitites A, B, C, D *are proportional, they are uſually wrote thus,* A : B : : C : D ; *and read thus, as* A *is to* B, *ſo is* C *to* D. *But when three quantities* A, B, C *are proportional, the middle one is repeated, and they are wrote thus,* A : B : : B : C.

4. Of three proportional quantities, the middle one is ſaid to be a Mean-proportional between the other two; and the laſt, a Third-proportional to the firſt and ſecond.

5. Of four proportional quantities, the laſt is ſaid to be a Fourth-proportional to the other three, taken in order.

A,B,C,D, E. 1. 2. 4. 8. 16.

6. Quantities are ſaid to be continually proportional (or in continual proportion) when the firſt is to the ſecond, as the ſecond to the third, as the third to the fourth, as the fourth to the fifth, and ſo on.

7. In a ſeries, or rank of quantities continually proportional, the ratio of the firſt and third, is ſaid to be duplicate to that of the firſt and ſecond ; and the ratio of the firſt and fourth, triplicate to that of the firſt and ſecond.

8. Any number of quantities, A, B, C, D being given, or propounded, the ratio of the firſt (A) to the laſt (D) is ſaid to be compounded of the ratios of the firſt to the ſecond, of the ſecond to the third, and ſo on to the laſt.

9. Ratio of equality, is that which equal quantities bear to each other.

It may be obſerved here, that ratio of equality, and equality of ratios, are, by no means, ſynonymous terms: ſince

ſince two, or more ratios may be equal, though the quantities compared are all unequal. Thus, the ratio of 2 *to* 1, *is equal to the ratio of* 6 *to* 3, (2 *being the double of* 1, *and* 6 *the double of* 3); *yet none of the four numbers are equal.*

10. Inverſe ratio is, when the antecedent is made the conſequent, and the conſequent the antecedent. *Thus, if* 2 : 1 : : 6 : 3; *then, inverſly,* 1 : 2 : : 3 : 6.

11. Alternate proportion is, when antecedent is compared with antecedent, and conſequent with conſequent.

As, if 2 : 1 : : 6 : 3; *then,* by alternation (or permutation) *it will be* 2 : 6 : : 1 : 3.

12. Compounded ratio is, when the antecedent and conſequent, taken as one quantity, are compared, either with the conſequent, or with the antecedent.

Thus, if 2 : 1 : : 6 : 3; *then,* by compoſition, $2+1 : 1 :: 6+3 : 3$, *and* $2+1 : 2 :: 6+3 : 6$.

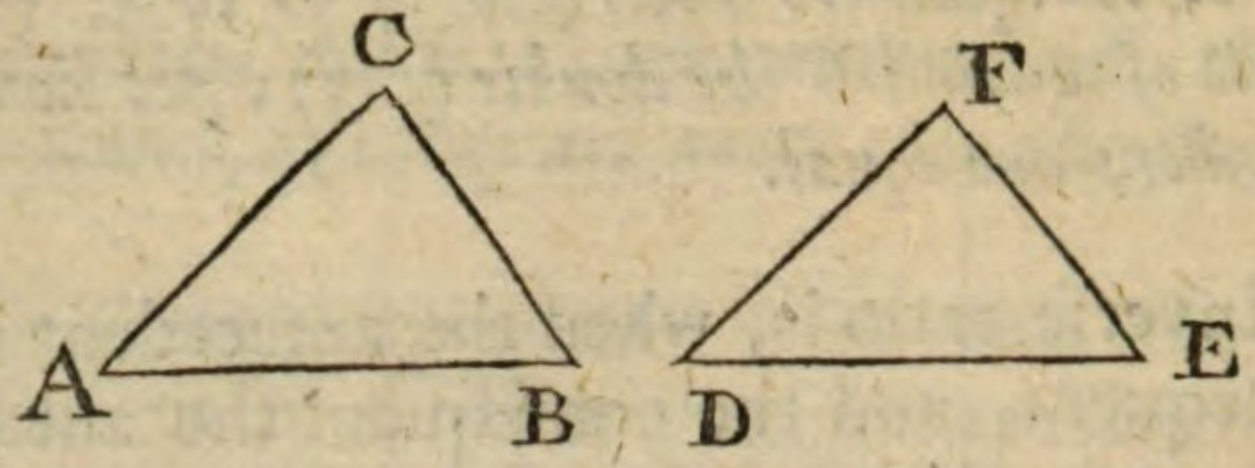
13. Divided ratio is, when the difference of the antecedent and conſequent is compared, either with the conſequent, or with the antecedent.

Thus, if 3 : 1 : : 12 : 4; *then,* by diviſion, $3-1 : 1 :: 12-4 : 4$, and $3-1 : 3 :: 12-4 : 12$.

Theſe four laſt definitions, which explain the names given by Geometers *to the different ways of managing and diverſifying of proportions, are put down here for the ſake of order; but are not to be uſed, or referred to, in any ſhape, till thoſe properties and relations are demonſtrated; which is effected in the three firſt Theorems of this book.*

14. Similar (or like) right-lined figures are ſuch, which have all their angles equal, one to another reſpec-

reſpectively, and alſo the ſides about the equal angles proportional.



Thus, if the angle A = D, B = E, C = F; alſo AC : AB : : DF : DE, BA : BC : : ED : EF, *&c.* then the figures ABC, DEF are ſaid to be *ſimilar*.

AXIOMS.

1. The ſame quantity being compared with ever ſo many equal quantities, ſucceſſively, will have the ſame ratio to them all.

2. Equal quantities, have to one and the ſame quantity, the ſame ratio.

3. Quantities having the ſame ratio to one and the ſame quantity, or to equal quantities, are equal among themſelves.

4. Quantities, to which one and the ſame quantity has the ſame ratio, are equal.

5. If two quantities be referred to a third, that which is the greateſt will have the greateſt ratio.

6. If two quantities be referred to a third, that is the greateſt which has the greateſt ratio.

7. Ratios, equal to one and the ſame ratio, are alſo equal, one to the other.

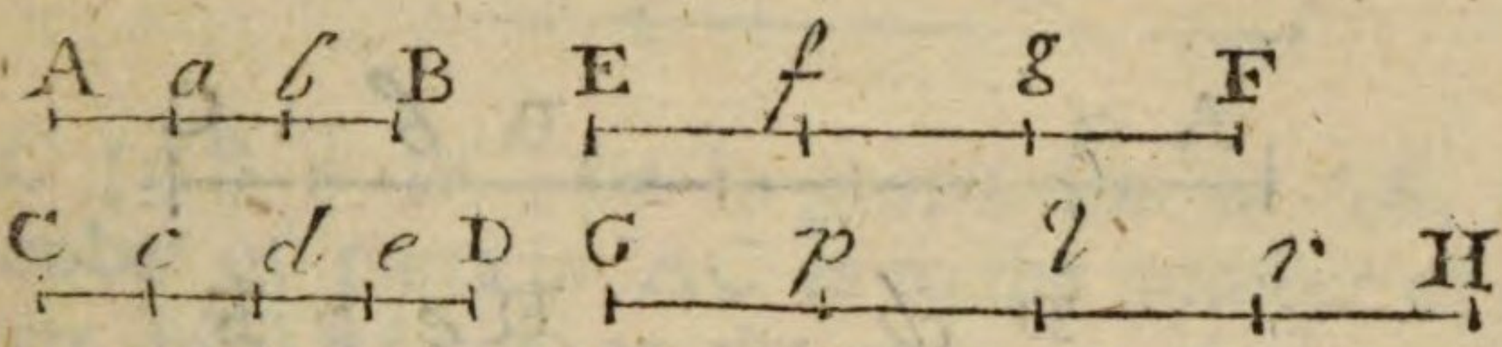
8. If two quantities be divided into, or compoſed of parts, that are equal among themſelves, or all of the ſame magnitude; then will the whole of the one, have the ſame ratio to the whole of the other, as the number of the parts in the one, has to the number of equal parts in the other.

9. If the double, treble, or quadruple, *&c.* of every part of any quantity be taken, the aggregate will

will be the double, treble, or quadruple, &c. of the whole quantity propounded.

THEOREM I.

Equimultiples of any two quantities (AB, CD) *are in the ſame ratio as the quantities themſelves.*



Let the ratio of AB to CD be that of any one number *M* (3) to any other number *N* (4), or, which is the ſame, let AB contain *M* (3) ſuch equal parts (A*a*, *ab*, *b*B[a]), whereof CD contains the number *N* (4). Let there be taken E*f*, *fg*, *g*F any equimultiples of A*a*, *ab*, *b*B, reſpectively; and let G*p*, *pq*, *qr*, *r*H be the ſame multiples of C*c*, *cd*, *de*, *e*D: ſo ſhall the whole EF be the ſame aſſigned multiple of the whole AB, and the whole GH of the whole CD, as each part in the one, is of its correſpondent in the other[b]. And, ſince the parts A*a*, *ab*, *b*B, C*c*, *cd*, &c. are all equal[c], their equimultiples E*f*, *fg*, *g*F, G*p*, *pq*, &c. will alſo be equal[d]. Therefore EF is in proportion to GH, as the number of the parts in EF is to the number of equal parts in GH[a], or (which is the ſame) as the number of parts in AB to the number of parts in CD, that is, as AB is to CD[a]: which was to be demonſtrated.

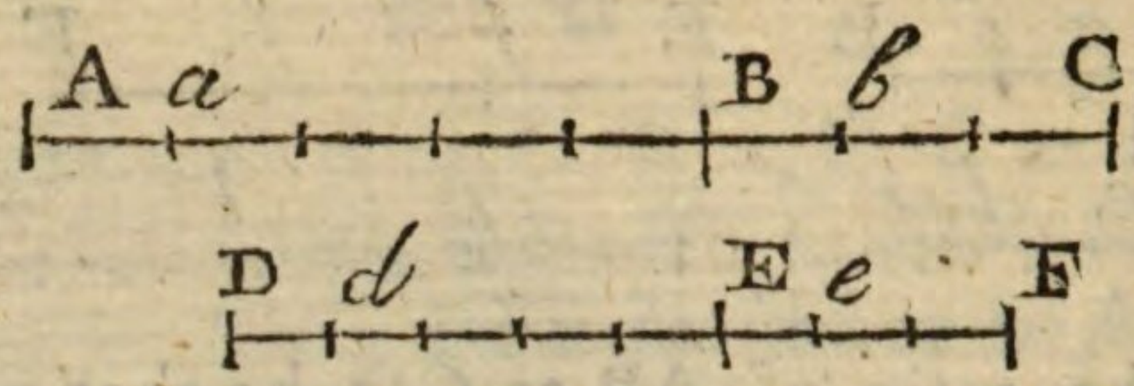
[a] Ax. 8. 4.
[b] Ax. 9. 4.
[c] Hyp.
[d] Ax. 4. 1.

COROLLARY.

Hence, like parts of quantities, have the ſame ratio as the wholes; becauſe the wholes are equimultiples of the like parts.

THEOREM II.

The two antecedents (AB, DE) *of four proportional quantities, of the ſame kind* (AB, BC, DE, EF) *are in the ſame ratio with the two conſequents* (BC, EF).



Let the common ratio of AB to BC, and of DE to EF, be that of any one number *M* (5) to any other number *N* (3); then will AB contain *M* (5) ſuch equal parts (A*a*) whereof BC contains [e] *N* (3); and DE will, in like manner, contain *M* (5) ſuch equal parts (D*d*), whereof EF contains *N* (3). And ſo, AB and DE, as well as BC and EF, being equimultiples of A*a* and D*d*, thence will AB : DE : : A*a* (B*b*) : D*d* (E*e*) : : BC : EF [f].

[e] Ax. 8. of 4.

[f] 1. 4.

COROLLARY.

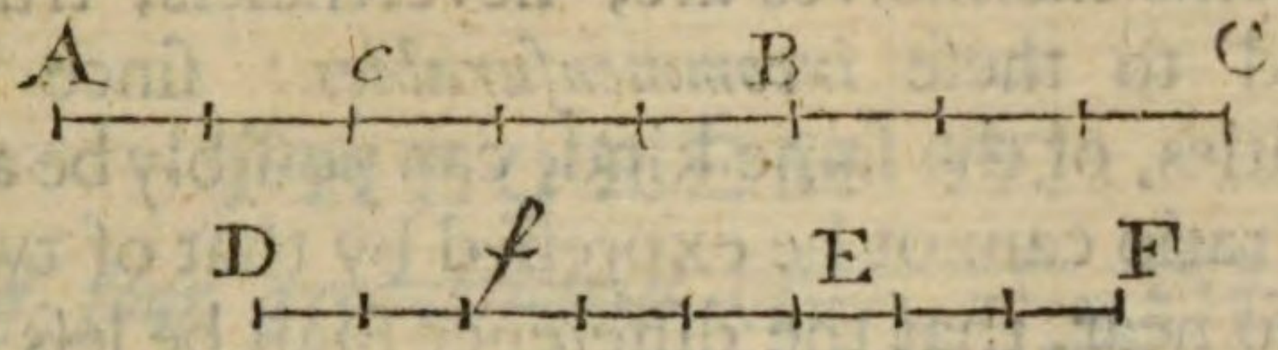
That, the proportionality will ſubſiſt, when the conſequents are taken as antecedents, and the antecedents as conſequents, alſo appears from hence. For BC : AB : : number of parts in BC (or EF) : number of parts in AB (or DE) : : EF : DE [e].

THEOREM III.

Of four proportional quantities (AB, BC, DE, EF) *the ſum, or difference of the firſt antecedent and conſequent* (AB $\pm$ BC) *is to the firſt antecedent, or conſequent, as the ſum, or difference of the ſecond antecedent and conſequent* (DE $\pm$ EF) *is to the ſecond antecedent, or conſequent.*

Let

Let what was premiſed in the demonſtration of the preceding theorem, be retained here: then will



AC (AB + BC) be in proportion to AB, as the number of parts in AC is to the number of equal parts in AB [g], or as the number of parts in DF [g] Ax. 8. 4. (DE + EF) to the number of equal parts in DE, that is, as DF (DE + EF) is to DE [g]. Again, if from AB and DE, be taken away B*c* = BC, and E*f* = EF, then will the difference A*c* be in proportion to AB, as the number of parts in A*c* (or D*f*) is to the number of parts in AB (or DE [g]), that is, as D*f* is to DE. In the ſame manner it will appear, that AB + BC : BC : : DE + EF : EF; and AB — BC : BC : : DE — EF : EF.

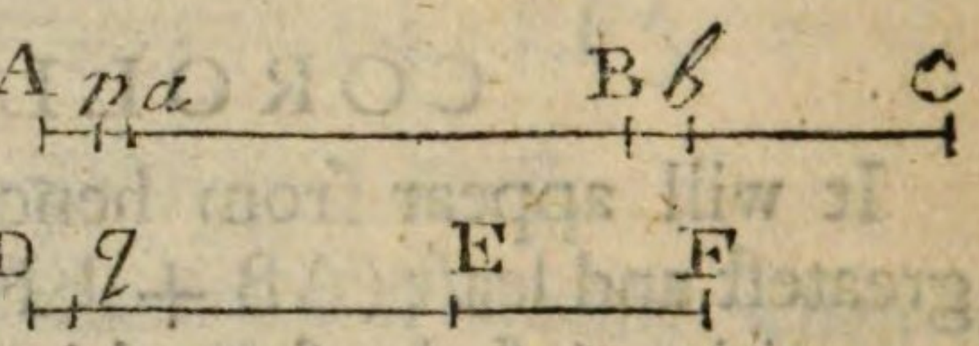
COROLLARY.

It will appear from hence, that the ſum of the greateſt and leaſt (AB + EF) of four proportional quantities (of the ſame kind) will exceed the ſum (BC + DE) of the two means: becauſe, AB being ſuppoſed greater than DE, A*c* will be greater than D*f*, in the ſame proportion [h]: and, if to theſe [h] 3. 4. there be added BC + EF (common); then will the ſum A*c* + BC + EF (AB + EF) be alſo greater than the ſum D*f* + BC + EF [i] (DE + BC.) [i] Ax. 6. 1.

SCHOLIUM.

In the demonſtration of this, and the preceding theorems, the antecedent and conſequent are ſuppoſed to be divided into parts, all mutually equal among themſelves. But it is known to Mathematicians, that there are certain quantities, or mag-

nitudes that cannot poſſibly be divided in that manner, by means of a *common meaſure.* The Theorems themſelves are, nevertheleſs, true, when applied to theſe *incommenſurables:* ſince no two quantities, of the ſame kind, can poſſibly be aſſigned, whoſe ratio cannot be expreſſed by that of two numbers, ſo near, that the difference ſhall be leſs than the leaſt thing that can be named. But if the matter, viewed in this light, ſhould not appear ſufficiently *ſcientific*, and you will not (in the preceding theorem) allow the ratio of AC to BC, to be *exactly* the ſame with that of DF to EF, when AB, BC, and DE, EF, are incommenſurables; then let it, if poſſible, be as ſome quantity *a*C (leſs than AC) is to BC, ſo is DE to EF. Let B*b* be a part or meaſure of BC leſs than the difference (A*a*) between AB and *a*B; let B*p* be that multiple of B*b*, which leaſt exceeds B*a*, and let *q*E be to EF, as *p*B to BC.



It is evident, that *p*B is leſs than AB (becauſe *ap* ⊐ B*b* ⊐ A*a*); and that *q*E is alſo leſs than DE, becauſe the ratio of *q*E to EF, being equal to that of *p*B to BC[k], it muſt neceſſarily be leſs than that of AB to BC[l], or of DE to EF, and ſo *q*E leſs than DE[m].

[f] Hyp.
[k] A*a*
[l] Ax. 5. 4.
[m] Ax. 6. 4.

Now, if (as is ſuppoſed) the ratio of *a*C to BC can be = the ratio of DF to EF, it muſt, of conſequence, be greater the ratio of *q*F to EF[n], or (which is the [o] ſame) than the ratio of *p*C to BC[l], *which is impoſſible.* In like manner it will appear, that no quantity, that is greater than AC, can poſſibly be to BC, as DF is to EF. Therefore AC : BC :: DF : EF. And by the ſame kind of argumentation (authoriſed and adopted by *Euclid* himſelf, in

[n] Ax. 5. and 7. 4.
[o] Proof of 3. 4.

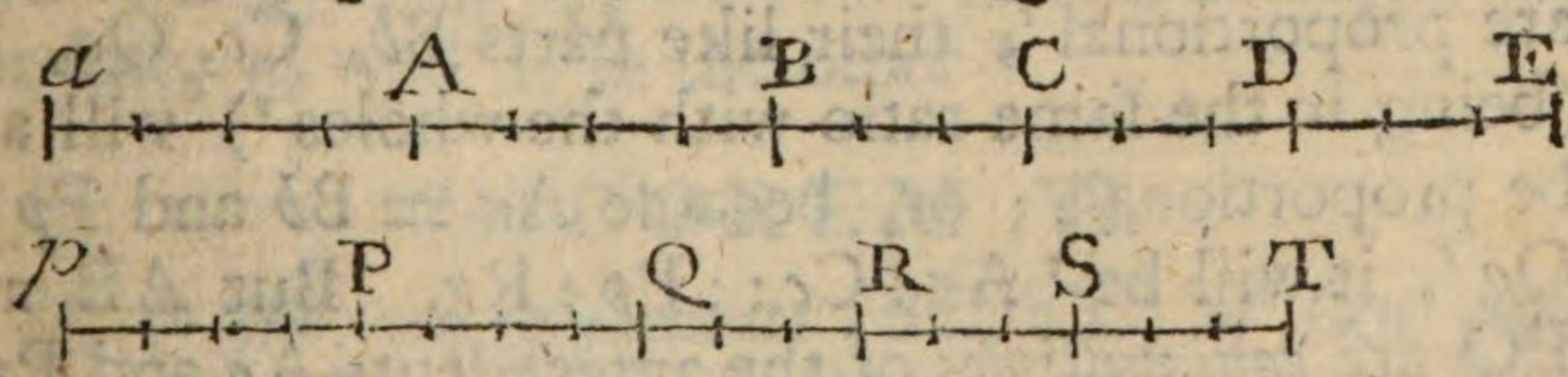
in his twelfth book) any difficulties, or ſcruples, that may be elſewhere brought, from the incommenſurability of quantities, may be obviated and removed.

THEOREM IV.

If, of four proportionals (AB, BC, PQ, QR) *equimultiples of the antecedents* (AB, PQ) *be taken, and compared with any equimultiples of their reſpective conſequents* (BC, QR), *the ratios will be the ſame, and the four quantities proportionals.*

Let the common ratio of AB to BC, and of PQ to QR, be that of any one number *M* to any other number *N*: ſo ſhall AB contain *M* ſuch equal parts whereof BC contains [p] *N*, and PQ, in like manner, *M* ſuch equal parts whereof QR contains *N*. [p] Ax.8. 4.

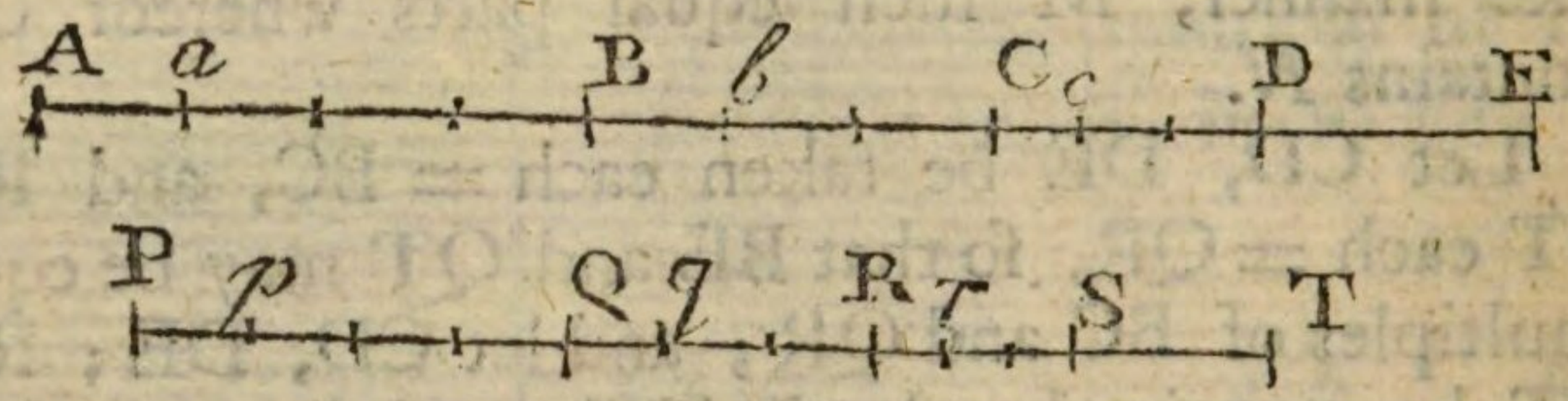
Let CD, DE be taken each = BC, and RS, ST each = QR, ſo that BE and QT may be equimultiples of BC and QR; and let CD, DE; RS, ST be conceived to be divided, each into the ſame number of parts with BC, or QR. In like manner, let *a*B and *p*Q be taken as equimultiples of AB and PQ, *&c.* Then will the number of parts in BD = number of parts in QS [p], and the number of parts in BE = number of parts in QT [q]: and ſo likewiſe with reſpect to *a*B and *p*Q. Therefore *a*B is to BE, as the number of parts in *a*B to the number of (equal) parts in BE [p], or, which is the ſame thing, as the number of parts in *p*Q to the number of parts in QT, that is, as *p*Q is to PT, *which was to be demonſtrated.* [q] Ax.4. 1.



THEOREM V.

If, of two ranks of quantities (AB, BC, CD; PQ, QR, RS), *the ratio of the firſt and ſecond, in the one, be equal to the ratio of the firſt and ſecond in the other, and the ratio of the ſecond and third, in the one, equal likewiſe to the ratio of the ſecond and third in the other; then, alſo, ſhall the ratio of the firſt to the third, be the ſame in the one rank, as in the other.*

Let the common ratio of AB to BC, and of PQ to QR, be ſtill expreſſed as in the preceding demonſtrations. Let, moreover, CD and RS be conceived to be divided, each into the ſame number of parts with BC and QR.



Becauſe the quantities BC, CD, QR, RS, [f Hyp. are proportional[r], their like parts B*b*, C*c*, Q*q*, R*r* [g Cor. to 1. 4. (being in the ſame ratio with the wholes[s]) will alſo be proportionals; or, becauſe A*a* = B*b* and P*p* = [t Ax. 2. 4. Q*q*[r], it will be[t] A*a* : C*c* : : P*p* : R*r*. But AB and PQ are *equimultiples* of the antecedents A*a* and P*p*[r]; and CD, RS are *equimultiples* of the conſequents [u 4. 4. C*c* and R*r*: therefore[u] AB : CD : : PQ : RS *; *which was to be demonſtrated.*

---

* *When, in two ranks of quantities, the proportions are* inordinate, *as* AB : BC : : QR : RS, *and* BC : CD : : PQ : QR; *the ſame thing may be demonſtrated; and that in the very ſame manner, except only, that* QR *muſt here be divided into the ſame number of parts with* AB, *and* PQ.

COROL-

COROLLARY I.

If other quantities DE, ST be taken, ſtill proportional to the two next preceding them, ſo that CD : DE : : RS : ST ; then, by the ſame argument (regard being had to AB, CD, DE in the one rank, and PQ, RS, ST in the other) it is evident, that AB : DE : : PQ : ST [x]. And thus we may go on, ſtill aſſuming other quantities, as many as we pleaſe ; and the ratio of the firſt and laſt, will always be the ſame in one rank, as in the other. Therefore ratios [y] compounded of the ſame number of like, or equal ratios, are equal.

[x] 5. 4.

[y] Def. 8. of 4.

COROLLARY II.

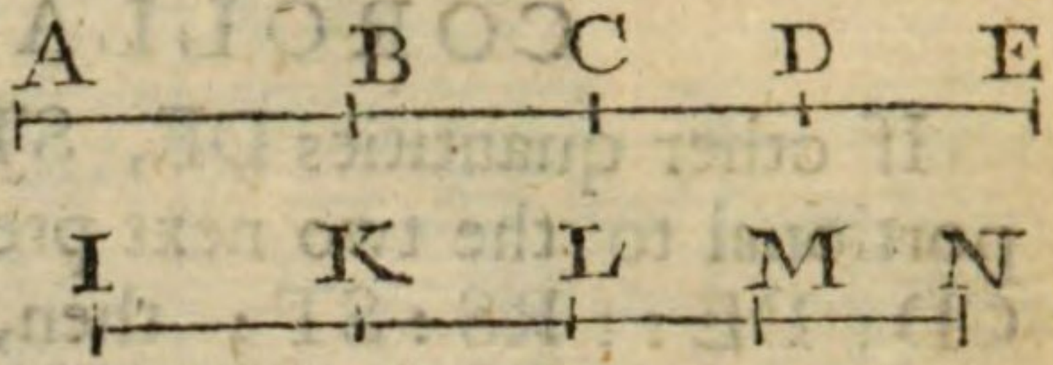
It is alſo evident from hence, that if any two quantities be taken proportional to the two conſequents of an aſſigned proportion, they will alſo be proportionals when compared with antecedents : and *vice verſâ*. For, the two quantities CD and RS, when compared, ſucceſſively, with the conſequents, and antecedents of the given proportion AB : BC : : PQ : QR, appear to be proportional, in the one caſe, as well as in the other [z].

[z] 5. 4.

THEOREM VI.

If, to the two conſequents (BC, KL) *of four proportionals* (AB, BC, IK, KL), *any two quantities* (CD, LM) *that have the ſame ratio to the reſpective antecedents be added ; theſe ſums and the antecedents will ſtill be proportionals (I ſay, if* AB : BC : : IK : KL, *and* AB : CD : : IK : LM ; *then ſhall* AB : BD : : IK : KM).

For,



For, CD and LM being proportional to the antecedents AB and IK [a], they and the consequents (BC, KL) will also be proportionals *(by Corol. 2. of the precedent)*: whence [b] *(by composition)* BC : BD : : KL : KM. And so again, *(by the same Corol.)* AB : BD : : IK : KM.

[a] Hyp.

[b] 3. 4.

COROLLARY.

From this Theorem it will appear, that, if the ratios of the corresponding quantities of two ranks with respect to the two first, are the same in both ranks (AB : BC : : IK : KL, AB : CD : : IK : LM, *&c.*); then the ratio of all the quantities to the first, will also be the same in the one rank, as in the other. For, by adding DE and MN to the last consequents (BD, KM) there will be had [c] AB : BE : : IK : KN (and so on, to any number of quantities whatever.) Then (by composition) AB : AE : : IK : IN.

[c] 6. 4.

When the quantities, in both ranks, are of the same kind, it will appear (by alternation) that the ratio of the two sums, and *that* of every two corresponding terms, will be the same.

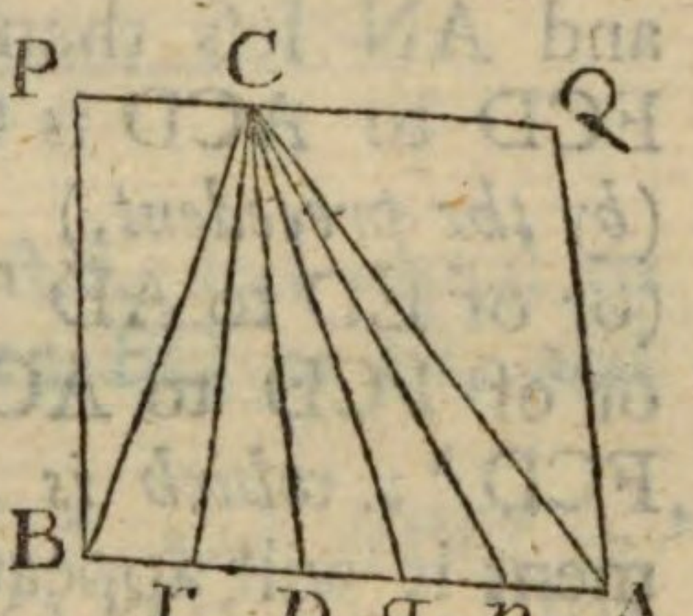
The six Theorems here delivered, on proportions of magnitudes in general, comprehend all that is most useful in that subject.—What relates to the proportions of extended magnitudes, under different limitations, and figures, as far as regards right lines and surfaces, will be the subject of the remaining part of this book.

Note,

Note, when-ever, in any demonſtration, you meet with ſeveral proportional quantities, connected continually by the ſign : : *(like theſe,* A : B : : C : D : : E : F : : G : H) *the concluſion to be drawn, is always from the firſt and laſt of the two equal ratios.*

THEOREM VII.

Triangles (ACD, BCD), *and alſo parallelograms* (ADCQ, BDCP), *having the ſame altitude, are to one another in the ſame ratio as their baſes* (AD, BD).



Let the baſe AD be to the baſe BD in the ratio of any one number *m* (3) to any other number *n* (2), or, which is the ſame[c], let AD contain *m* (3) ſuch equal parts whereof BD contains the number *n* (2). Then, the triangles AC*p*, *p*C*q*, BC*r*, *&c.* made by drawing lines from the points of diviſion to the vertex C, being all equal among themſelves[d]; the triangle ACD will therefore be in proportion to the triangle BCD, as the number of equal parts in the former to the number of equal parts in the latter, or as the number of parts in AD to the number of parts in BD, that is, as AD to BD[e]. Whence, alſo, the parallelograms ADCQ, BDCP, being the doubles of their reſpective triangles[d], are likewiſe in the ſame ratio as their baſes AD and BD[e].

[c] Ax. 8. 4.

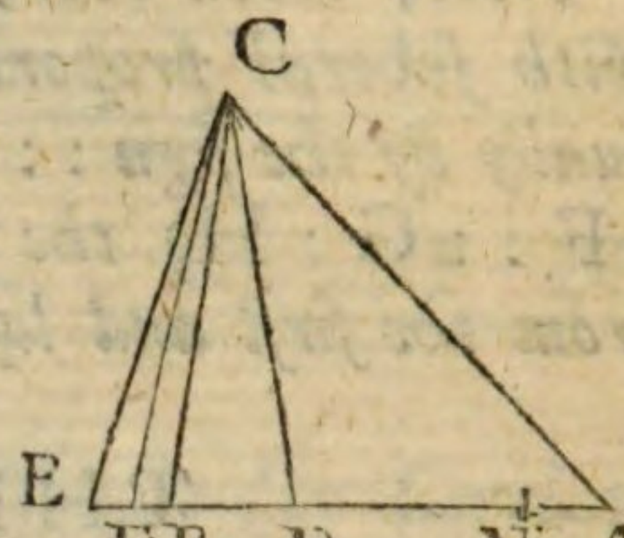
[d] Cor. 2. to 2. 2.

[e] 4. 4.

SCHOLIUM.

If the baſes AD *and* BD *are incommenſurable to each other, the ratio of the triangles cannot be other than that of their baſes.*

For,



For, if poſſible, let the triangle BCD be to the triangle ACD, not as BD to AD, but as ſome other line ED, greater than BD, is to AD.

Let AN be a part, or measure of AD, leſs than BE [c], and let DF be that multiple of AN which leaſt exceeds DB; alſo let CE and CF be drawn. It is manifeſt that the point F falls between B and E, becauſe *(by Hyp.)* BF is leſs than AN, and AN leſs than BE. Moreover, the ratio of FCD to ACD is the ſame as that of FD to AD *(by the precedent.)* But the ratio of BCD to ACD (or of ED to AD [f]) is ⌐ the ratio of FD to AD [g], or of FCD to ACD; and conſequently BCD ⌐ FCD [h]: *which is impoſſible* [i]. By the ſame argument it will appear, that the triangle BCD cannot be to the triangle ACD, as a line, leſs than BD, is to AD. Therefore BCD : ACD : : BD : AD.

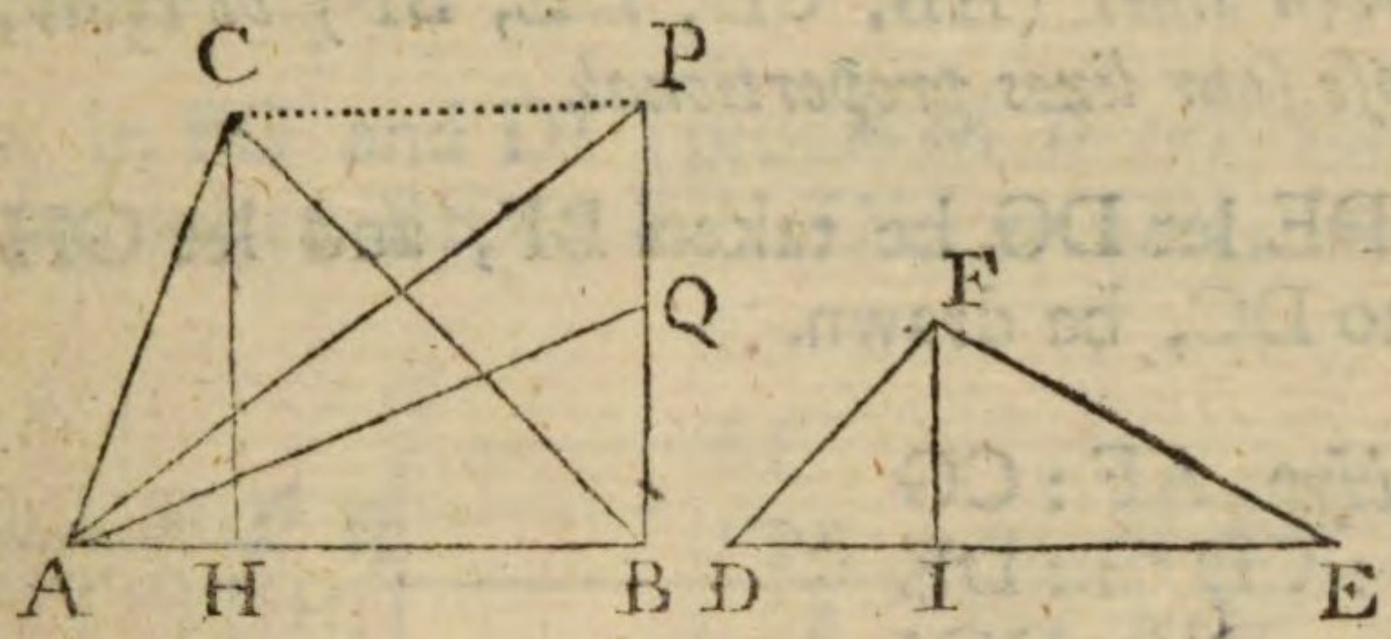
[c] Lem. 1. 8.
[f] Hyp.
[g] Ax. 5. of 4.
[h] Ax. 6. of 4.
[i] Ax. 2. of 5.

If this Scholium ſhould appear difficult to the Learner, it may not be amiſs to omit it intirely; ſince it is only put down for the ſake of thoſe who may be ſcrupulous about the buſineſs of incommenſurables; to whom it may not be improper to obſerve, that nothing more is taken for granted herein, than what is effected by means of the firſt Lemma in the 8th book; which being demonſtrated from axioms, and one ſingle theorem in the firſt book, is referred to here, though not given till hereafter, for reaſons already hinted at, in this note.

THEO-

THEOREM VIII.

Triangles (ABC, DEF) *ſtanding upon equal baſes* (AB, DE) *are to one another, in the ſame ratio as their altitudes* (CH, FI.)



Let BP be perpendicular to AB, and equal to CH; in which let there be taken BQ = FI, and let AP and AQ be drawn.

The triangle ABP is = ABC [k], and ABQ = DEF [k]; but ABP (ABC) : ABQ (DEF) : : [1] BP (HC) : BQ (FI).

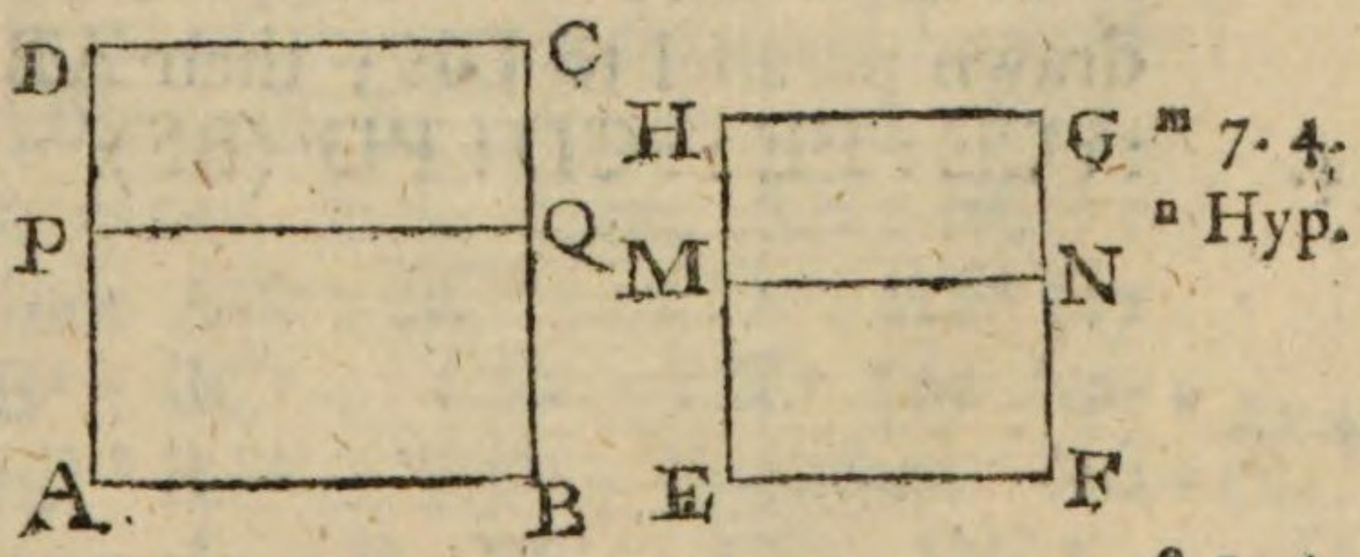
[k] Cor. 2. to 2. 2.

[1] 7. 4.

THEOREM IX.

If, parallel to the baſes of any two parallelograms (AC, EG), *two lines* (PQ, MN) *be drawn, ſo as to cut the ſides proportionally* (AP : AD : : EM : EH), *then will thoſe parallelograms and their correſponding parts* (AQ, EN) *be alſo proportionals.*

For, AQ : AC : : AP : AD [m] : : EM : EH [n] : : EN : EG : [m]; and therefore, *by alternation*, AQ : EN : : AC : EG [o],



[m] 7. 4.

[n] Hyp.

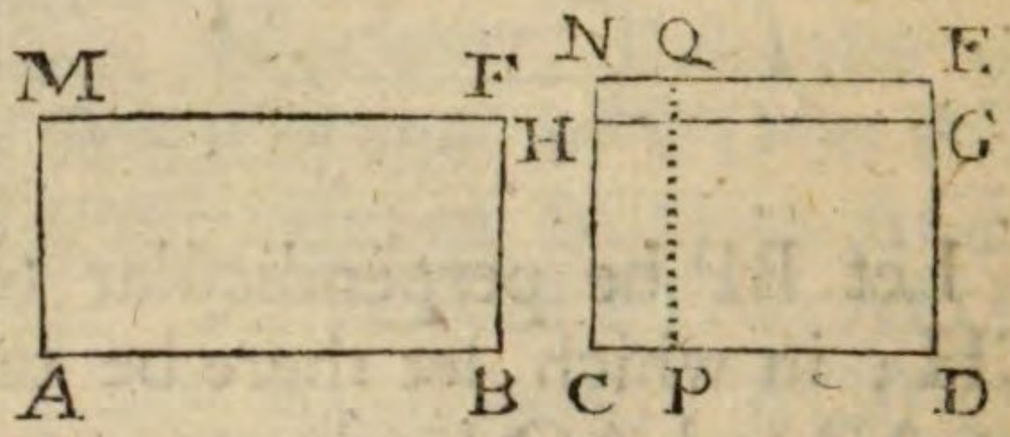
[o] 2. 4.

THEO-

THEOREM X.

If four lines are proportional (AB : CD : : DE : BF), *the rectangle* (AF) *under the two extremes will be equal to the rectangle* (CE) *under the two means. And, if the rectangles under the extremes and means of four given lines* (AB, CD, DE, BF) *be equal, then are thoſe four lines proportional.*

In DE let DG be taken BF, and let GH, parallel to DC, be drawn.



1. *Hyp.* AF : CG
[p] 7. 4. : : AB : CD [p] : : DE
[q] Hyp. : BF [q] : : DE : DG [r]
[r] Ax. 1. 4. : : CE : CG [p]; therefore, the conſequents of the firſt and laſt of theſe equal ratios being the ſame quantity CG, the two antecedents AF and CE muſt be equal [s].

[s] Ax. 3.

[t] Ax. 2. 4. 2. *Hyp.* AB : CD : : AF : CG [p] : : CE : CG [t] : : DE : DG [p] (BF.)

SCHOLIUM.

From the ſame demonſtration, and ſcheme, it will appear, that the two antecedents of four proportional lines (AB, CD, DE, BF) are in the ſame ratio to each other, as the two conſequents: for, if in DC there be taken DP = BF, and PQ be drawn parallel to DE; then AB : DE : : AF : PE [t] : : CE : PE : : CD : PD (BF).

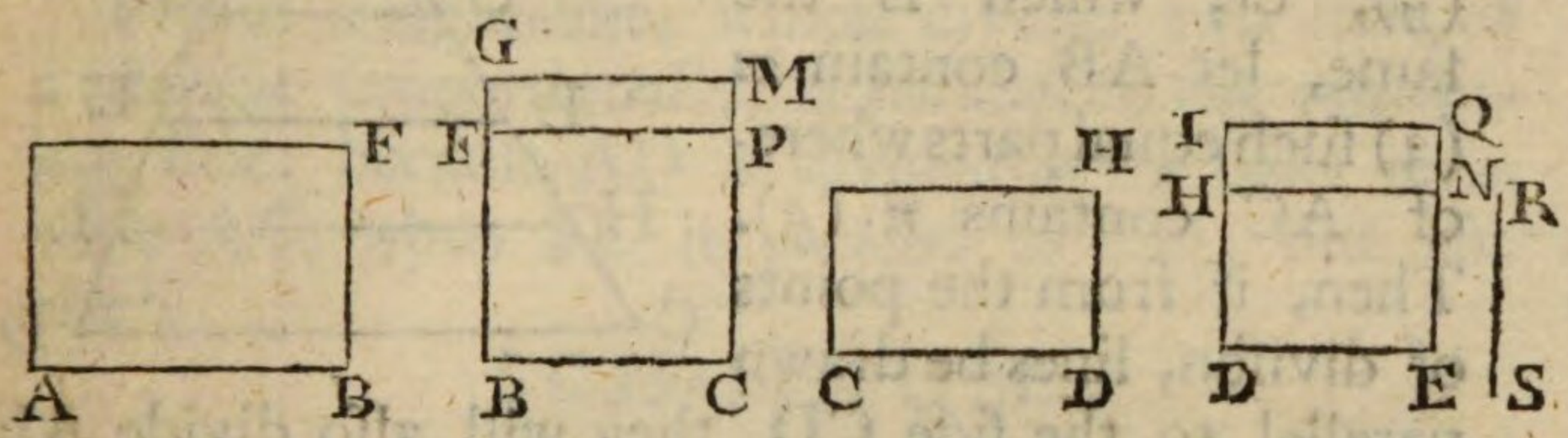
[t] 7. 4.

THEO-

THEOREM XI.

The rectangles under the corresponding lines, of two ranks of proportionals, are themselves proportionals. (*I say, if* AB : BC : : CD : DE, *and* BF : BG : : DH : DI, *then will the rectang.* AF : *rectang.* BM : : CH : *rect.* DQ).

For, in BG and DI (produced if neceſſary) let there be taken BF = BF, DH = DH, and let



FP be parallel to BC, and HN to DE: then AF : BP : : AB : BC [y] : : CD : DE [z] : : CH : DN [y]; whence *(alternately)* AF : CH : : PB : DN, and ſo likewiſe is BM to DQ [a]: whence *(again by alternation)* AF : BM : CH : DQ.

[y] 7. 4.
[z] Hyp.
[a] 9. 4.

COROLLARY I.

Hence, the ſquares of four proportional lines, are themſelves proportional.

COROLLARY II.

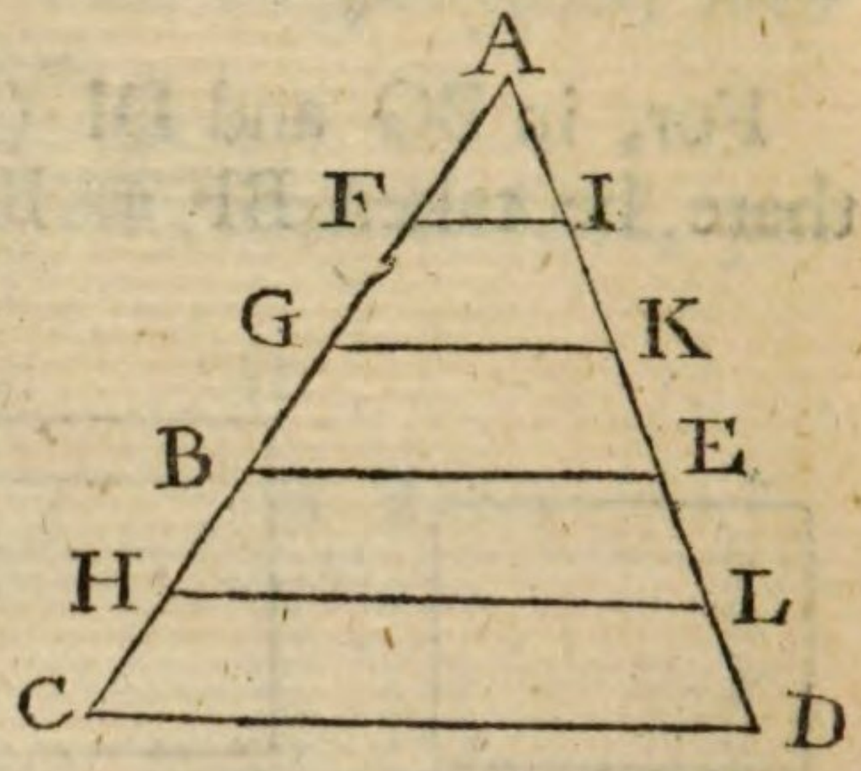
Hence alſo, the ſides of four proportional ſquares (AB^2, BC^2, CD^2, DE^2) will be proportional. For, let the line RS be taken ſuch, that, AB : BC : : CD : RS; then, ſince $AB^2 : BC^2 :: CD^2 : RS^2$ (*by Corol.* 1.) and $AB^2 : BC^2 :: CD^2 : DE^2$ (*by ſuppoſition*): thence will [b] $RS^2 = DE^2$; therefore [c] RS = DE, and conſequently AB : BC : : CD : DE [d].

[b] Ax. 4. 4.
[c] Cor. 2. of 6. 2.
[d] Ax. 1. 4.

THEO-

THEOREM XII.

A line (BE) *drawn parallel to one ſide* (CD) *of a triangle* (ACD) *divides the other two ſides proportionally.* (I ſay, AB : AC : : AE : AD, AB : BC : : AE : ED, and AC : BC : : AD : ED).



Let AB be to AC, as any one number *m* (3) is to any other number *n* (5), or, which is the ſame, let AB contain *m* (3) ſuch equal parts whereof AC contains *n* (5). Then, if from the points of diviſon, lines be drawn parallel to the ſide CD, they will alſo divide AE and AD into the like numbers of equal parts [e]. Therefore AE is to AD, as the number of equal parts in AE to the number of equal parts in AD, or as the number of equal parts in AB to the number of equal parts of AC, that is, as AB to AC [f]. In the ſame manner, AE is to ED, as the number of parts in AE to the number of parts in ED, or as the number of parts in AB to the number of parts in BC, that is, as AB to BC. Alſo, in the ſame manner, AC : BC : : AD : ED.

[e] Cor. 1. to 27. 1.

[f] Ax. 8.

The ſame otherwiſe.

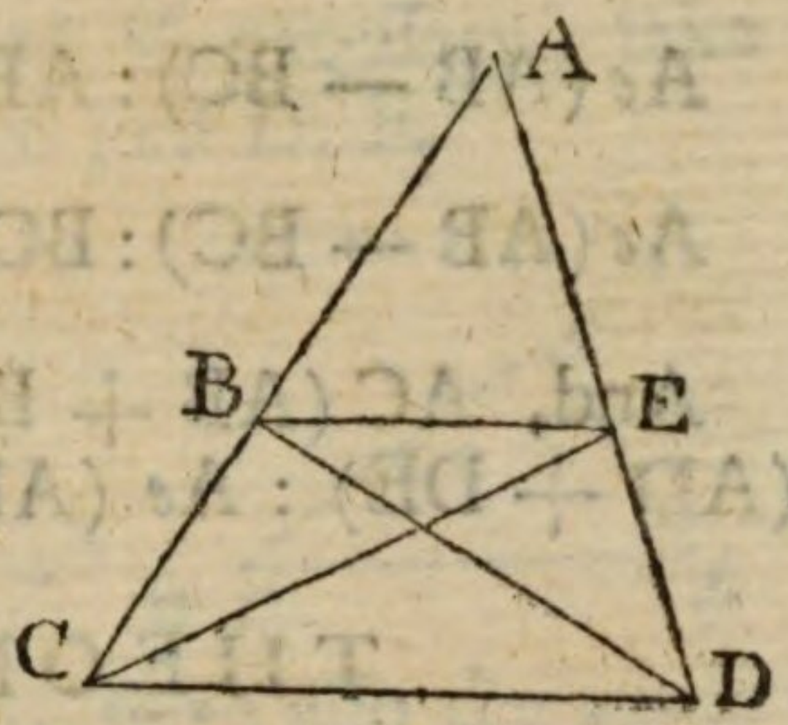
Draw CE and DB. Then will the triangles BEC and EBD be equal to each other [f]; whence, by adding BEA to both, AEC will be alſo = ABD [g].

[f] Cor. 1. to 2. 2.

[g] Ax. 4. 1.

But,

But, AB : AC : : triang. AEB : triangle AEC [h] (ABD) : : AE : AD [h]; and AB : BC : : triang. AEB : triangle CEB [h] (DEB) : : AE : ED.



[h] 7. 4.

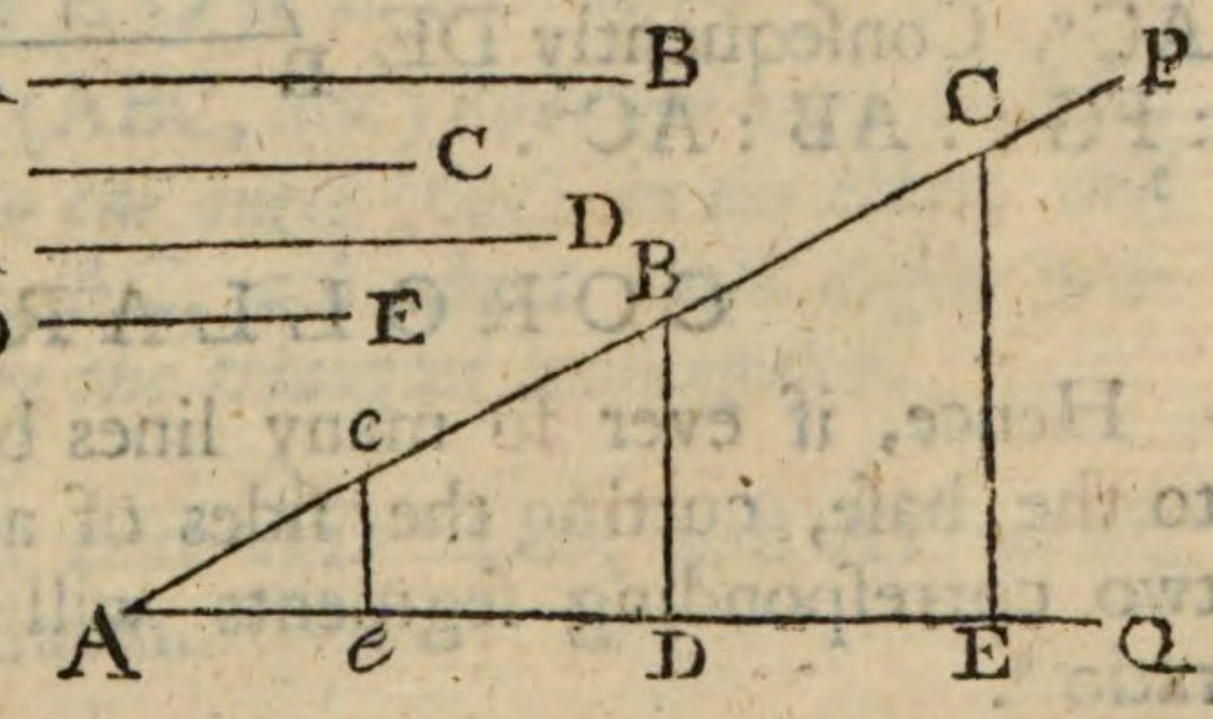
COROLLARY.

Hence a right-line, which divides two ſides of a triangle proportionally, is parallel to the remaining ſide: becauſe AD is divided in the ſame ratio with AC, when BE is parallel to CD; but not elſe [i]. [i] Ax. 2. 1. and 5. 4.

SCHOLIUM.

From this laſt Theorem, whatever relates to the compoſition and diviſion of ratios, when theſe reſpect the compariſon of right-lines, will appear exceedingly obvious.

For, let AB, BC, AD, and DE, be proportionals: and from any point A, let two indefinite right-lines AP, AQ be drawn; in which take AB = AB, BC = BC, AD = AD, and DE = DE; alſo take B*c* = BC, D*e* = DE, and let BD, CE, and *ce* be drawn.



Since AB : BC : : AD : DE [k], thence is EC parallel to DB [l]; and ſo, B*c* being = BC, and D*e* = DE, *ec* will alſo be parallel to DB [m]. Therefore,

[k] Hyp.
[l] Cor. of 12. 4.
[m] Cor. 2. to 27. 1.

AC (AB + BC) : AB : : AE (AD + DE) : AD;
AC (AB + BC) : BC : : AE (AD + DE) : DE;

Ac (AB — BC) : AB : : Ae (AD — DE) : AD;

Ac (AB — BC) : BC : : Ae (AD — DE) : DE;

And, AC (AB + BC) : Ac (AB — BC) : : AE (AD + DE) : Ae (AD — DE).

THEOREM XIII.

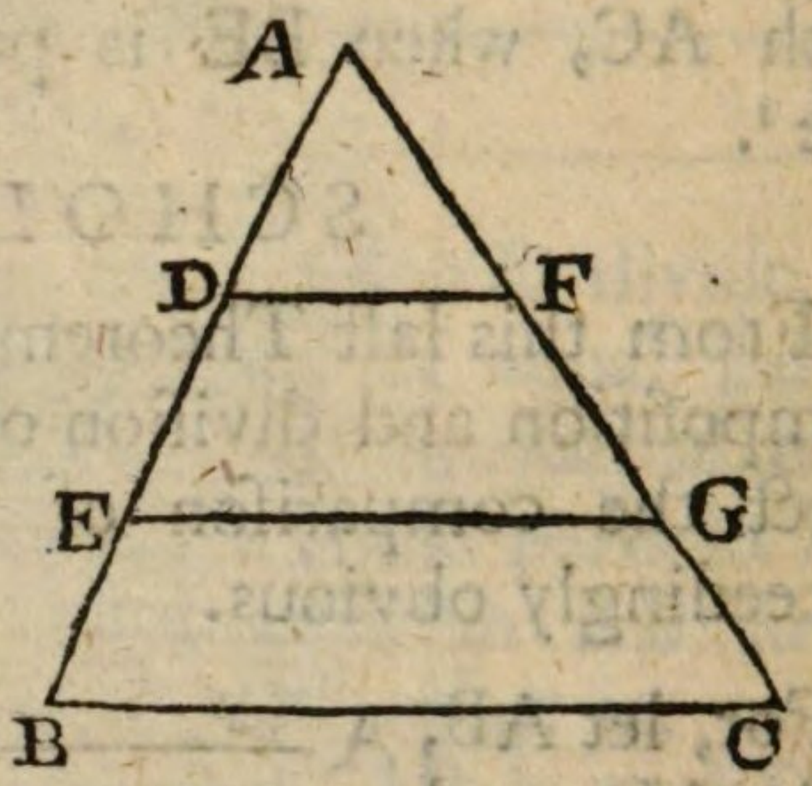
The parts (DE, FG) *of the two ſides of a triangle, intercepted by right-lines* (DF, EG) *drawn parallel to the baſe* (BC), *are in the ſame ratio with the wholes.* (DE : FG : : AB : AC).

For, DF and EG being parallel to each other, thence will DE : AE : : FG : AG [o]; therefore, (by alternation) DE : FG : : AE : AG [p]. In the ſame manner, AE : AG : : AB : AC [o]. Conſequently DE : FG : : AB : AC [q].

[o] 12. 4.

[p] 2. 4.

[q] Ax. 7. 4.



COROLLARY.

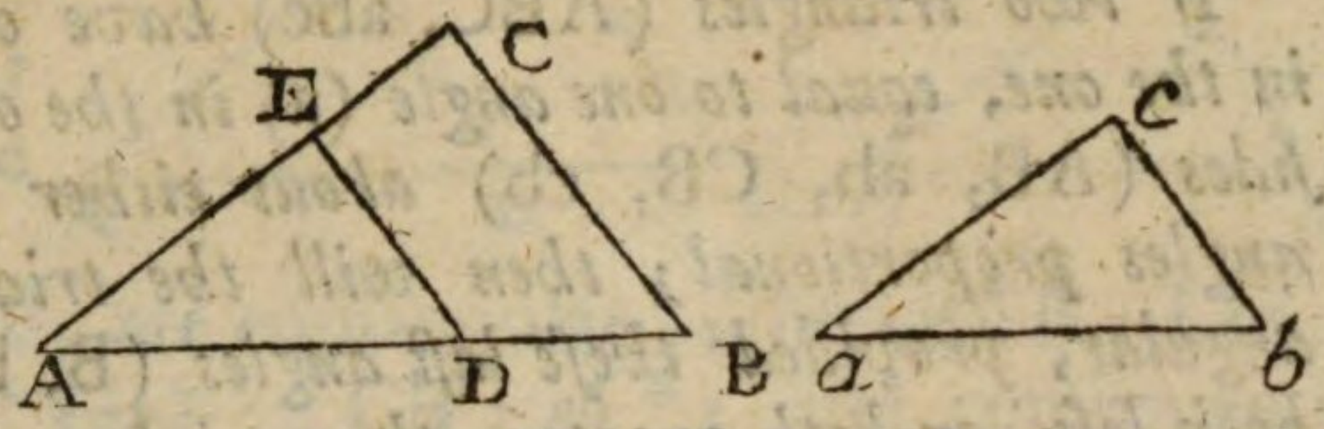
Hence, if ever ſo many lines be drawn parallel to the baſe, cutting the ſides of a triangle, every two correſponding ſegments will have the ſame ratio [q].

THEOREM XIV.

In triangles (ABC, abc) *mutually equiangular, the correſponding ſides* (AB, ab, AC, ac) *containing the equal angles* (A, a) *are proportional.*

In

In AB and AC (produced if neceſſary) take AD = *ab*, and AE = *ac*, and join D, E.



The triangles *abc*, and ADE, having *ab* = AD, *ac* = AE, and the angle *a* = A, have alſo the angle ADE [r] = *abc* = ABC [s]; whence DE will be parallel to BC [t]; and therefore AB : AD (*ab*) : : AC : AE [u] (*ac*).

[r] Ax. 10. 1.
[s] Hyp.
[t] Cor. to 8. 1.
[u] 12. 4.

The ſame otherwiſe.

Becauſe AB × *ac* = *ab* × AC [w], therefore is AB : *ab* : : AC : *ac* [x].

[w] 24. 3.
[x] 10. 4.

COROLLARY.

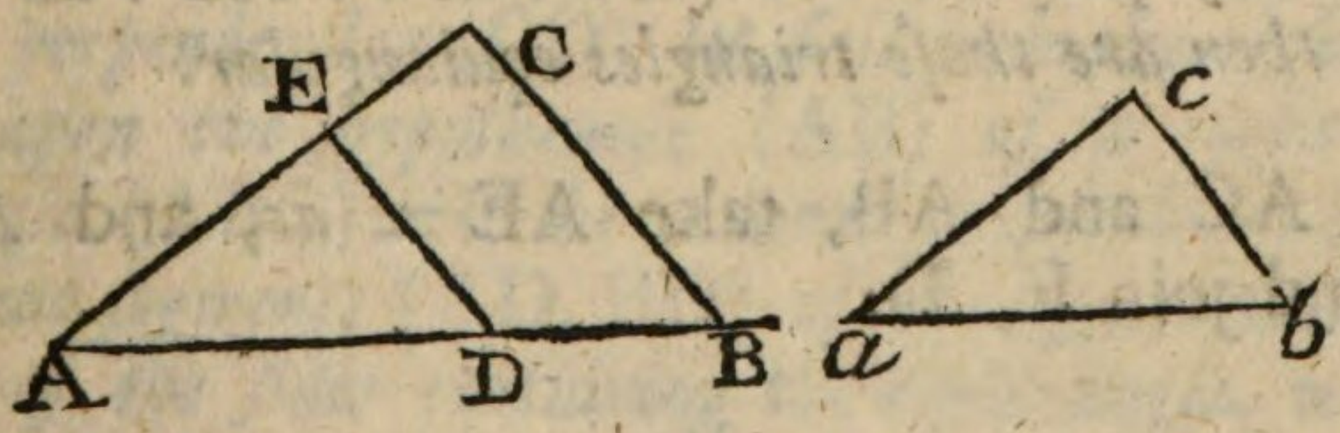
Hence equiangular triangles are ſimilar to each other [y].

[y] Def. 14. of 4.

THEOREM XV.

If two triangles (ABC, abc) *have one angle* (BAC) *in the one, equal to one angle* (bac) *in the other, and the ſides* (AB, ab, AC, ac) *about thoſe angles proportional; then are the triangles equiangular.*

In AB and AC, take AD = *ab*, and AE = *ac*, and let DE be drawn.



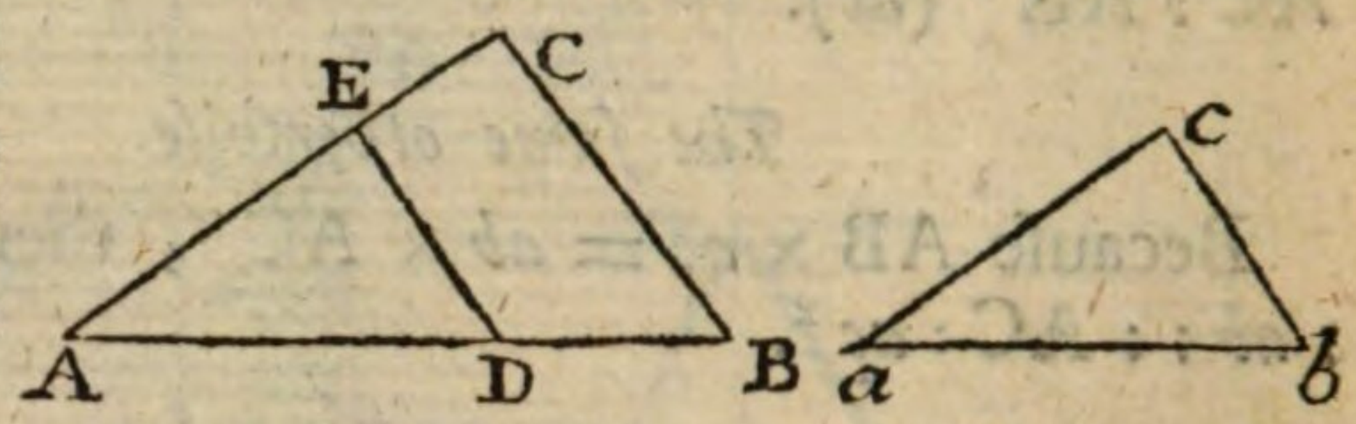
Since AB : *ab* (AD) : : AC : *ac* (AE [b]), therefore is DE parallel to BC [c]; whence the angle B = ADE [d] = *b* [e], and the angle C = AED = *c*.

[b] Hyp.
[c] Cor. to 12. 4.
[d] Cor. 1. to 7. 1.
[e] Ax. 10. 1.

THEOREM XVI.

If two triangles (ABC, abc) *have one angle* (A) *in the one, equal to one angle* (a) *in the other, and the ſides* (AB, ab, CB, cb) *about either of the other angles proportional; then will the triangles be equiangular, provided theſe laſt angles* (B, b) *be, either, both leſs, or both greater, than right-angles.*

In AB, let AD be taken = *ab*, and let DE be drawn parallel to BC, meeting AC in E.



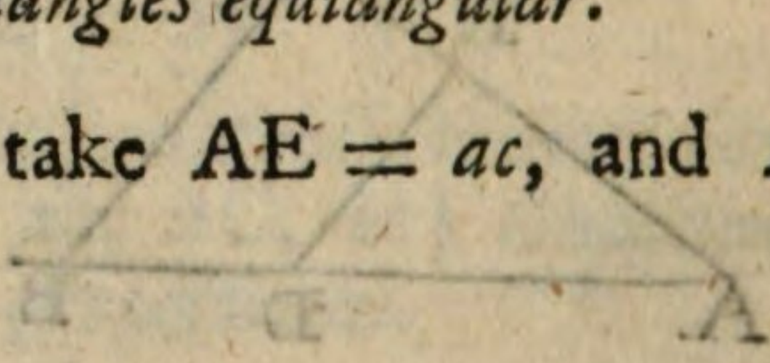
Then will the triangles ABC, and ADE, be equiangular [f]; therefore, CB : ED : : AB : AD [g] : : AB : *ab* [h] : : CB : *cb* [i]; and conſequently ED = *cb* [k]: whence the triangles *abc* and ADE (having *ab* = AD, *cb* = ED, and *a* = A) will be equal in all reſpects [l], provided the angles *abc* and ABC (= ADE) are either both leſs, or both greater than right-angles. Therefore, ſince the latter of theſe equal triangles (*abc*, ADE) is equiangular to ABC, the propoſition is manifeſt.

[f] Cor. 1. to 7. 1.
[g] 14. 4.
[h] Ax. 1. 4.
[i] Hyp.
[k] Ax. 4. 4.
[l] 17. 1.

THEOREM XVII.

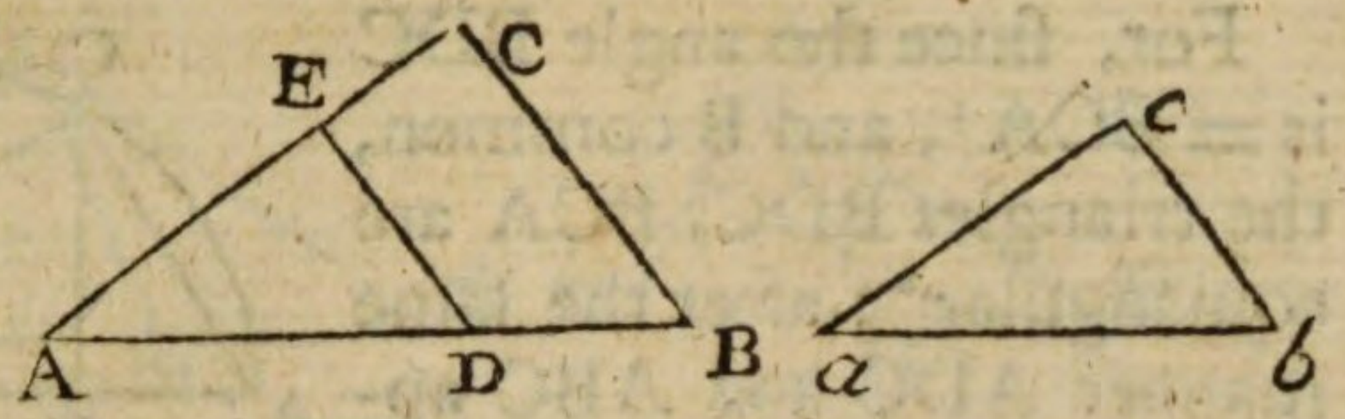
If two triangles (ABC, abc) *have all their ſides, reſpectively proportional* (AC : ac : : AB : ab : : CB : cb) *then are thoſe triangles equiangular.*

In AC and AB, take AE = *ac*, and AD = *ab*, and join E, D.



Since

Since AC ; AE (*ac*) : : AB : AD (*ab*) [m], the triangles



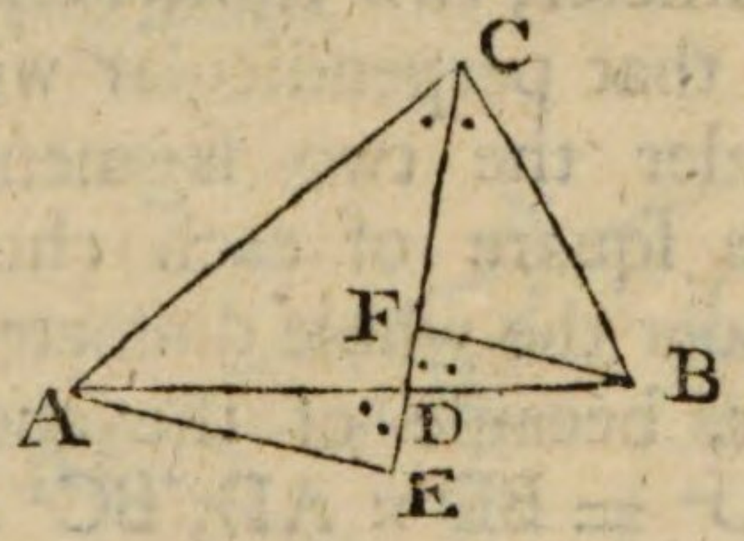
[m] Hyp.

ABC, ADE, are equiangular [n]; hence CB : ED : : [n] 15. 4. AB : AD [o] (*ab*) : : CB : *cb* [m]; and consequently [o] 14. 4. ED = *cb* [p]: therefore the triangles *abc*, ADE, [p] Ax. 4. 4. being mutually equilateral, they must also be mutually equiangular [q]; and consequently *abc*, as well [q] 14. 1. as ADE, equiangular to ABC.

THEOREM XVIII.

A right-line (CD) *bisecting any angle* (ACB) *of a triangle* (ABC) *divides the opposite side* (AB) *into two segments* (AD, BD) *having the same ratio with the sides* (AC, CB) *containing that angle.*

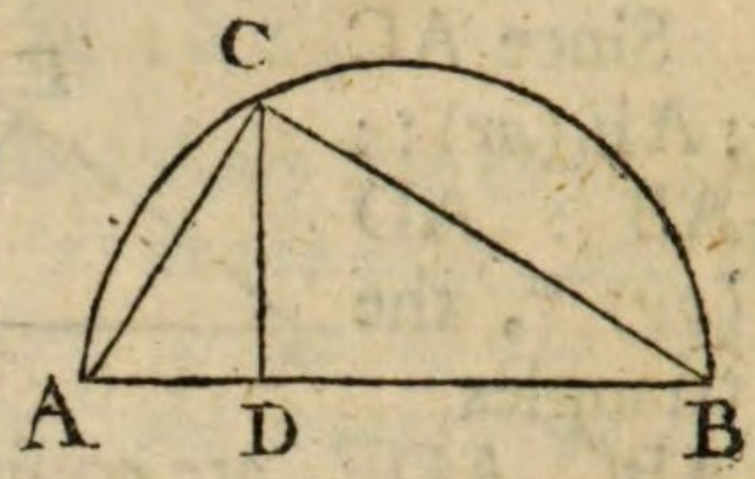
Let AE and BF be perpendicular to CDE. Then the triangles ACE, CBF, and ADE, BDF being, respectively, equiangular [r], it will be AD : BD : : AE : BF [s] : : AC : BC.



[r] Hyp. and 3. 1.
[s] 14. 4. and 2. 4.

THEOREM XIX.

A perpendicular (CD) *let fall from the right-angle* (C) *upon the hypothenuse* (AB) *of a right-angled triangle* (ABC) *will be a mean proportional between the two segments* (AD, BD) *of the hypothenuse: and each of the sides containing the right-angle, will be a mean proportional between its adjacent segment, and the whole hypothenuse.*



For, ſince the angle BDC is = BCA [t], and B common, the triangles BDC, BCA are equiangular [u]: after the ſame manner ADC and ABC appear to be equiangular.

[t] Ax. 7.

[u] Cor. 1. to 10. 1.

Therefore, by Theor. XIV,

BD : CD : : CD : AD
AB : BC : : BC : BD
AB : AC : : AC : AD.

COROLLARY.

Becauſe the angle in a ſemi-circle is a right-angle [w], it follows, that, if from any point C, in the periphery of a ſemi-circle ACB, a perpendicular CD be let fall upon the diameter AB, and from the ſame point C, to the extremities of that diameter, two chords CA, CB be drawn; the ſquare of that perpendicular will be equal to a rectangle under the two ſegments of the diameter; and the ſquare of each chord, equal to a rectangle under the whole diameter and its adjacent ſegment: for, becauſe of the above proportions, we have $CD^2 = BD \times AD$, $BC^2 = AB \times BD$, and $AC^2 = AB \times AD$.

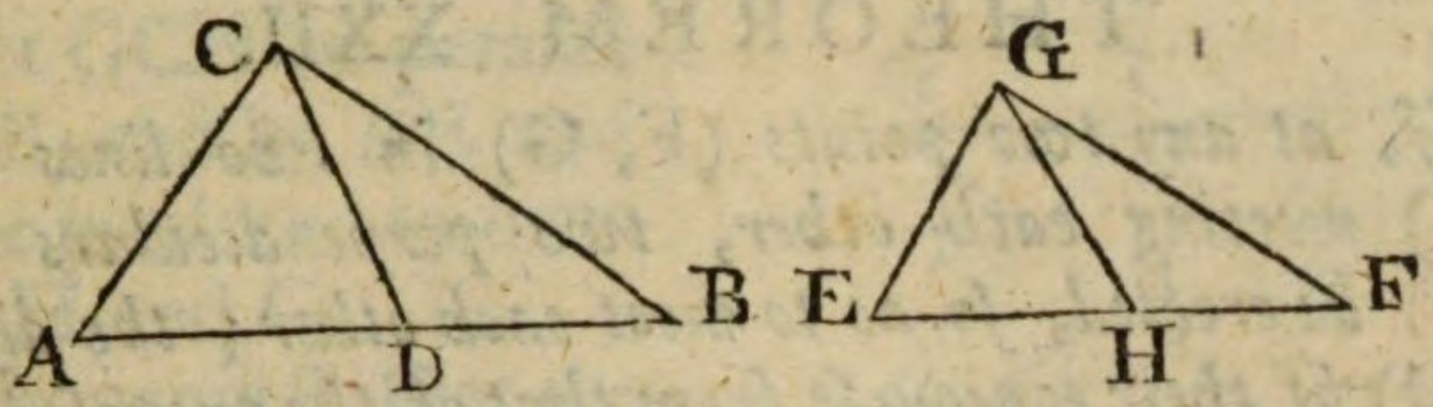
[w] 13. 3.

[x] 10. 4.

THEOREM XX.

If, in ſimilar triangles (ABC, EFG) *from any two equal angles* (ACB, EGF) *to the oppoſite ſides, two right-lines* (CD, GH) *be drawn, making equal angles with the homologous ſides* (CB, GF); *thoſe right-lines will have the ſame ratio as the ſides* (AB, EF) *on which they fall, and will alſo divide thoſe ſides proportionally.*

For,



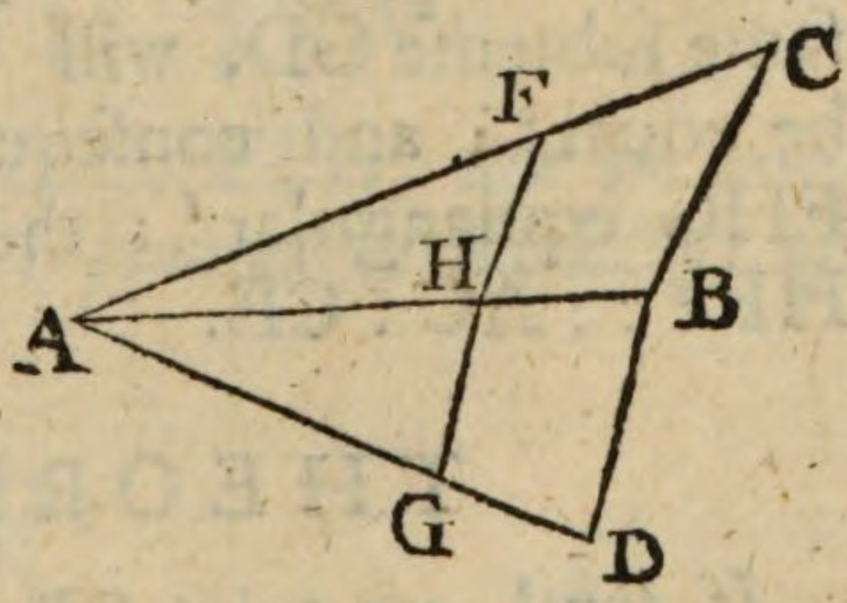
For, the triangles ADC, EHG, and BDC, FHG (as well as the *wholes* ABC, EFG) being equiangular [y],
thence is [z] AB : EF (:: AC : EG) :: CD : GH;
and AD : EH (:: DC : HG) :: BD : FH.

[y] Hyp.
[z] 14. 4. and Ax. 7.

THEOREM XXI.

If in two triangles (ABC, ABD) *having one side* (AB) *common to both, from any point* H *in that side, two lines* (HF, HG) *respectively parallel to two contiguous sides* (BC, BD) *be drawn, to terminate in the two remaining sides* (AC, AD); *those lines* (HF, HG) *will have the same ratio as the sides* (BC, BD) *to which they are parallel.*

For, AB : AH :: BC : HF [a], and AB : AH :: BD : HG [a]; therefore, by equality, BC : HF :: BD : HG; whence, alternately, BC : BD :: HF, HG [b].



[a] 14. 4.
[b] 2. 4.

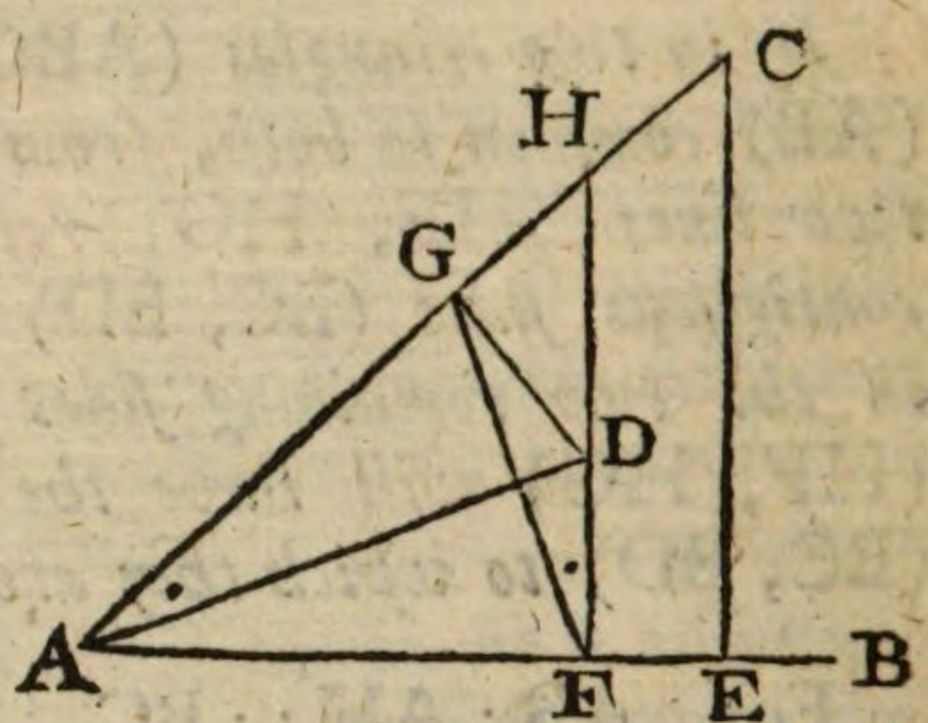
COROLLARY.

Hence, if BC = BD, then alſo will HF = HG.

THEOREM XXII.

If, at any two points (F, G) *in two lines* (AB, AC) *meeting each other, two perpendiculars* (FD, GD) *be erected, ſo as to meet each other; the diſtance* (AD) *of their concourſe from that of the propoſed lines, will be to the diſtance* (FG) *of the two points themſelves, in the ratio of one of the ſaid lines* (AC) *to a perpendicular* (CE) *falling from the extreme thereof upon the other* (AB).

Let FD be produced to meet AC in H.



Since the angles AFD and AGD are right-ones [c], the circumference of a circle will paſs thro' all the four points A, F, D, G [d]; and ſo the angles GFD, GAD, ſtanding on the ſame ſubtenſe GD, will be equal [e]; and conſequently the triangles AHD, FHG equiangular [f]: therefore AD : FG : : AH : HF [g] : : AC : CE.

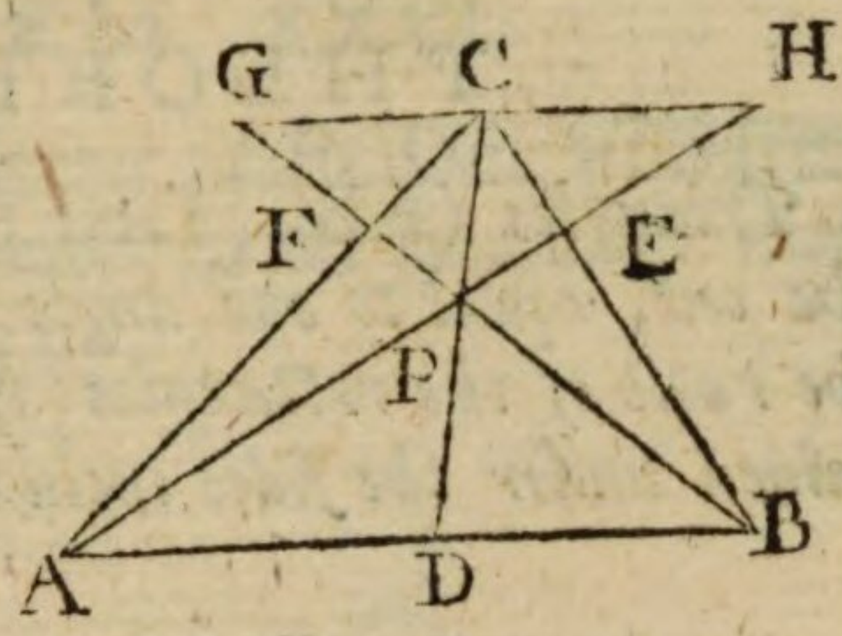
[c] Hyp.
[d] 19. 3.
[e] 11. 3.
[f] Cor. 1. to 10. 1.
[g] 14. 4.

THEOREM XXIII.

If thro' any point (P) *in a triangle* (ABC) *three right-lines* (AE, BF, CD) *be drawn, from the angular points to cut the oppoſite ſides, the ſegments* (AD, BD) *of any one ſide* (AB) *will be to each other, as the rectangles* (AF × CE, BE × CF) *under the ſegments of the other ſides taken alternately.*

Let

Let GCH be parallel to AB, and let AE and BF be produced to meet it in H and G.



It is manifeſt, that the triangles FBA, FCG; EAB, EHC; APB, GPH, are equiangular[h]:

[h] 7. and 3. of 1.

Therefore AF : CF : : AB : CG[i], CE : BE : : CH : AB[i].

[i] 14. 4.

Whence AF x CE : CF x BE : : CH x AB : CG x AB[k] : : CH : CG[l]; but CH : CG : : AD : BD[m]; therefore, by equality, AF x CE : CF x BE : : AD : BD.

[k] 11. 4.
[l] 7. 4.
[m] 20. 4.

COROLLARY.

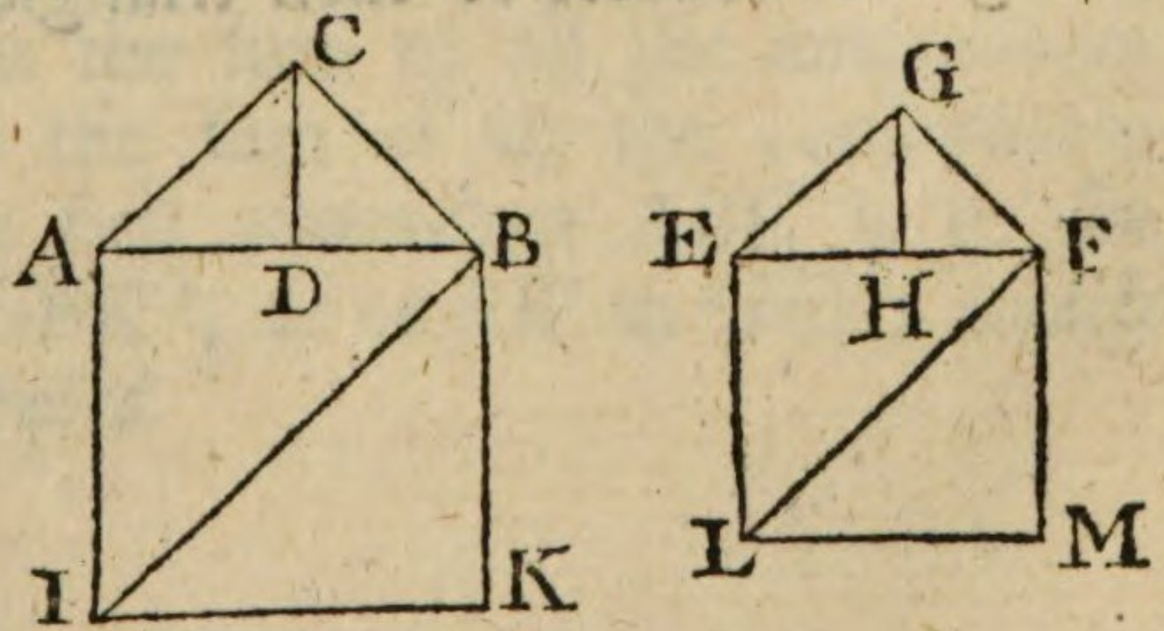
Hence, if AD = BD, then alſo will AF x CE = CF x BE, and therefore AF : CF : : BE : CE[n].

[n] 10. 4.

THEOREM XXIV.

Equiangular triangles (ABC, EFG) *are in proportion to one another, as the ſquares* (AK, EM) *of their homologous ſides.*

Upon AB and EF let fall the perpendiculars CD and GH, and let the diagonals BI, FL be drawn.



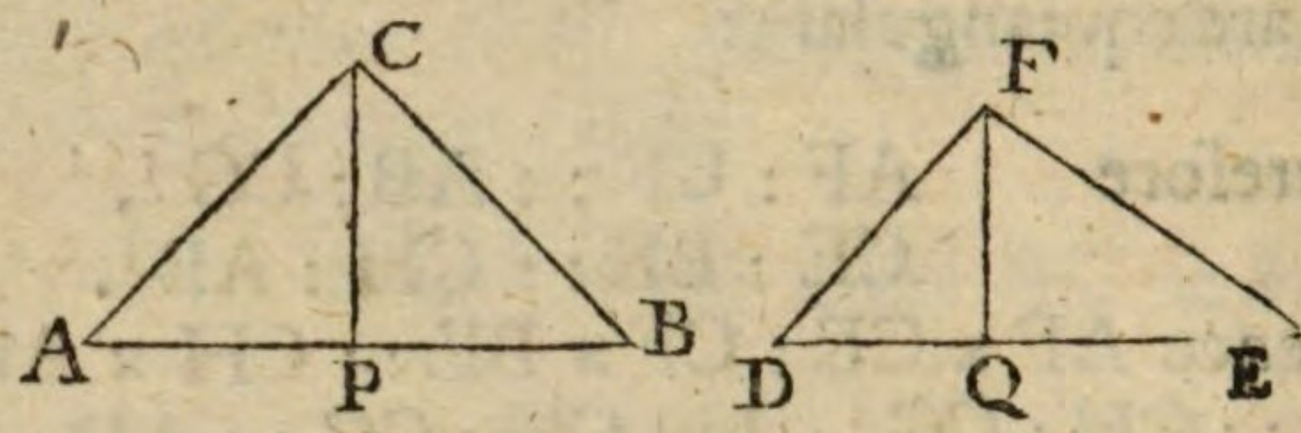
Becauſe ABC : ABI : : CD : AI (AB)[o] : : GH : EF (EL) : : EFG : EFL[o]; therefore, alternately, ABC : EFG : : ABI : EFL[q] : : AK : EM[r].

[o] 8. 4.
[p] 20. 4.
[q] 2. 4.
[r] Cor. 2. to 2. 2.

THEO-

THEOREM XXV.

Triangles (ABC, DEF) *having one angle* (A) *in the one, equal to one angle* (D) *in the other, are in the ratio of the rectangles* (AC × AB, DF × DE) *contained under the sides including the equal angles.*



Upon AB and DE, let fall the perpendiculars CP and FQ. Then AC × AB : CP × AB : : AC : CP [s] : : DF : FQ [t] : : DF × DE : FQ × DE [s]; whence, alternately, AC × AB : DF × DE : : CP × AB : FQ × DE : : triangle ABC : triangle DEF [u].

[s] 7. 4.
[t] 14. 4.
[u] Cor. 1. 4.

COROLLARY.

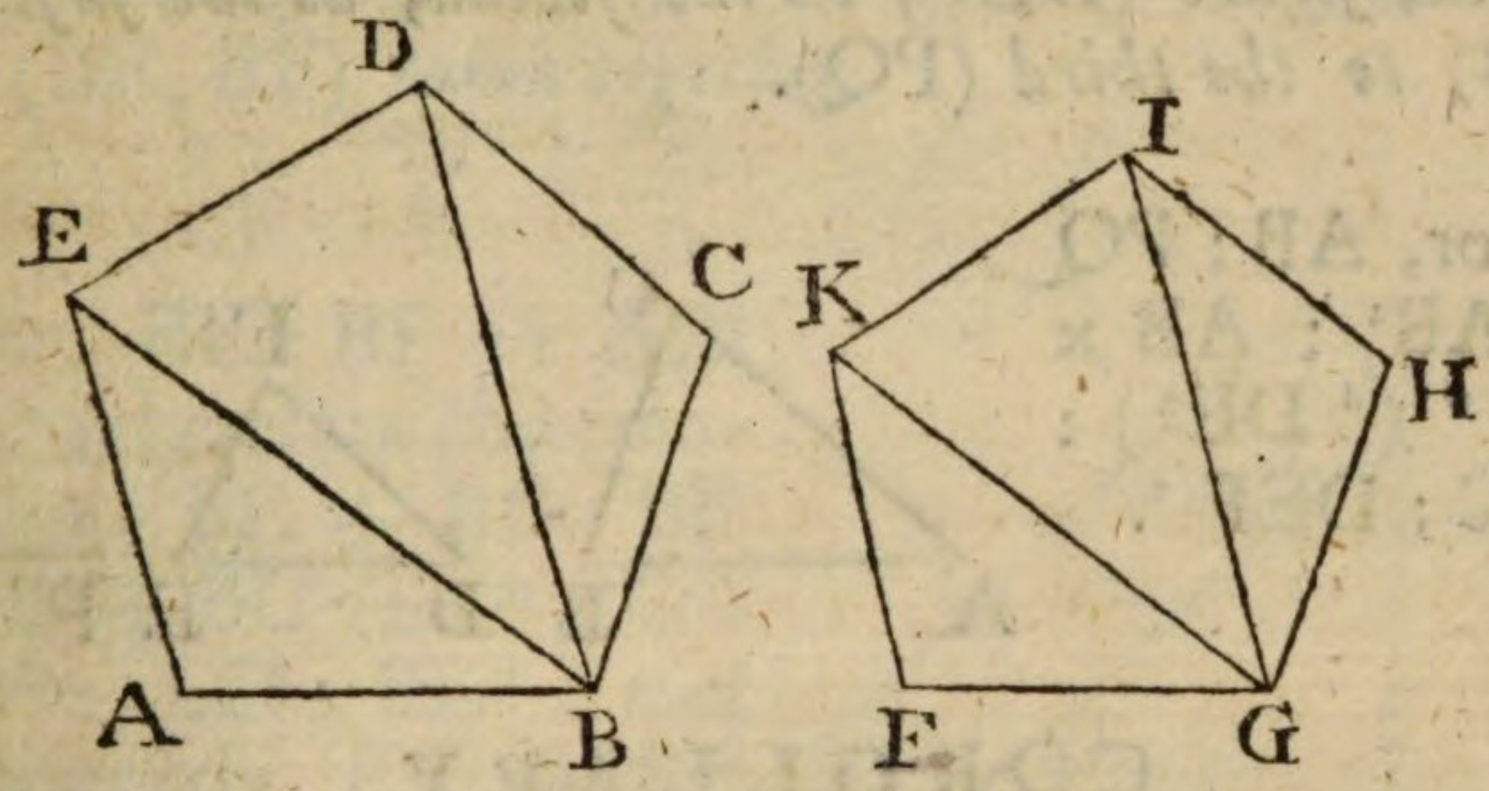
Hence, if the rectangles of the sides containing the equal angles, be equal, or the sides themselves reciprocally proportional [w], the triangles will be equal. The same also holds in parallelograms, being the doubles of such triangles.

[w] 10. 4.

THEO-

THEOREM XXVI.

All ſimilar right-lined figures (ABCDE, FGHIK) *are in proportion to one another as the ſquares of their homologous ſides* (AB, FG).



Draw the right-lines BE, BD, GK, GI.

Becauſe A = F, and AB : AE : : FG : FK [x], the triangles BAE, GFK are equiangular [y]; therefore, if from AED = FKI [x], there be taken AEB = FKG, the remainders BED, GKI will alſo be equal [z]. Wherefore, ſince ED : KI (: : EA : KF [x]) : : EB : KG [w], the triangles EBD, KGI are likewiſe equiangular [y]. In the ſame manner it will appear, that DBC, IGH are alſo equiangular.

[x] Def. 14. of 4.
[y] 15. 4.
[z] Ax. 5. of 1.
[w] 14. 4.

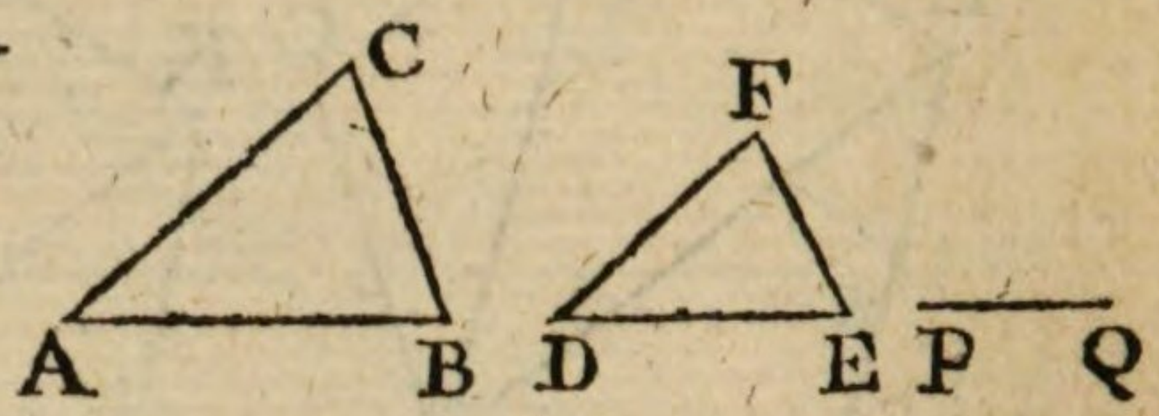
Therefore, becauſe ABE : GFK (: : [a] BE² : GK²) : : BED : GKI (: : [a] BD² : GI²) : : [a] BDC : GIH, it is evident, that the ſum of all the antecedents (ABCDE) is to the ſum of all the conſequents (FGHIK) as the firſt antecedent ABE is to the firſt conſequent GFK [b], or as AB² to [a] FG²; *which was to be demonſtrated.*

[a] 24. 4.
[b] Cor. to 6. 4.

THEO-

THEOREM XXVII.

If three right-lines (AB, DE, PQ) *are proportionals, the right-lined figure* (ABC) *upon the first, will be in proportion to the similar, and similarly described, figure* (DEF) *on the second, as the first line* (AB) *to the third* (PQ).



For, AB : PQ
: : AB^2 : AB × PQ [c] ([d] DE^2) : ABC : DEF [e].

[c] 7. 4.
[d] Hyp. and 10. of 4.
[e] 26. 4.

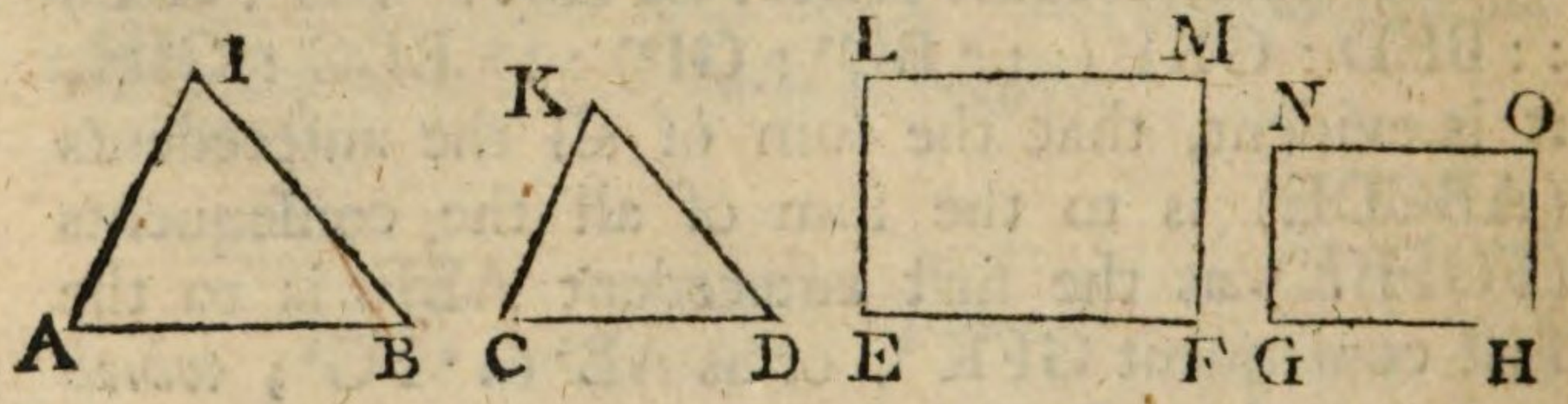
COROLLARY.

Hence, similar right-lined figures, are in the duplicate ratio of their homologous sides [f].

[f] Def. 7. of 4.

THEOREM XXVIII.

If four right-lines (AB, CD, EF, GH) *be proportional, the right-lined figures described upon them, being like, and in like sort situate, shall also be proportional* (ABI : CDK : : EM : GO).



For, ABI : CDK (: : [g] AB^2 CD^2 : : EF^2 : GH^2 [h])
: : EM : GO [g].

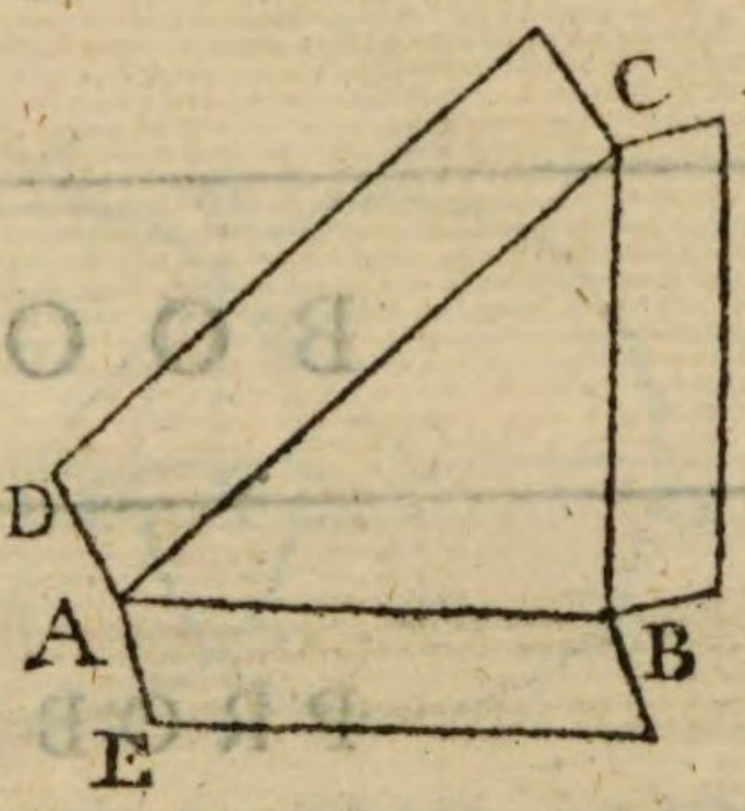
[g] 26. 4.
[h] Cor. 1. to 11. 4.

THEO-

THEOREM XXIX.

If upon the three ſides of a right-angled triangle (ABC) *as many right-lined figures* (CD, BE, BF) *like, and alike ſituate, be deſcribed, that* (CD) *upon the hypothenuſe* (AC) *will be equal to both the other two* (BE, BF) *taken together.*

For, BE : BF : : AB^2 : [l] BC^2; therefore (by compoſition) BE + BF : BE : : $AB^2 + BC^2$ (= [m] AC^2) : AB^2 : : CD : BE [l]; and conſequently BE + BF = CD [n].



[l] 26. 4.

[m] 8. 2.

[n] Ax. 3. of 4.

The End of the FOURTH BOOK.

ELEMENTS

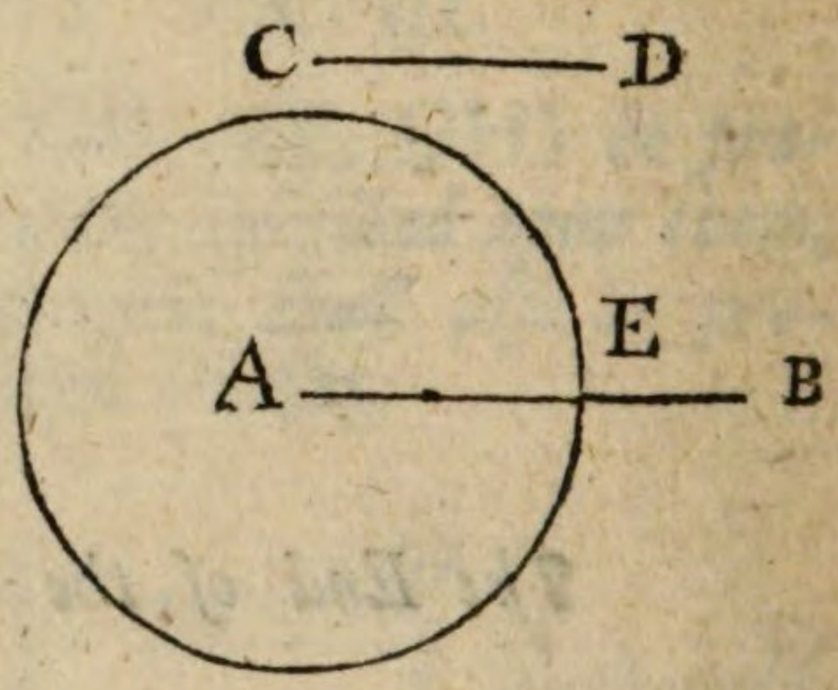
OF

GEOMETRY.

BOOK V.

PROBLEM I.

FROM the greater (AB) *of two unequal lines* (AB, CD) *to cut off, or take away a part* (AE) *equal to the leſſer* (CD).



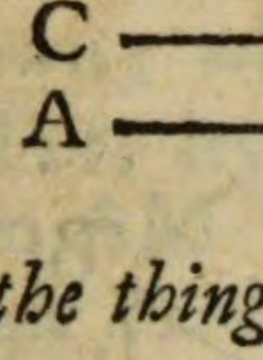
From A as a center, with a radius equal to CD, let the circumference of a circle be deſcribed [a], cutting [b] AB in E; *and the thing is done.*

[a] Poſt. 3.
[b] Ax. 2.

PROBLEM II.

At a given point (A) *to make a line* (AB) *equal to a given line* (CD).

Draw

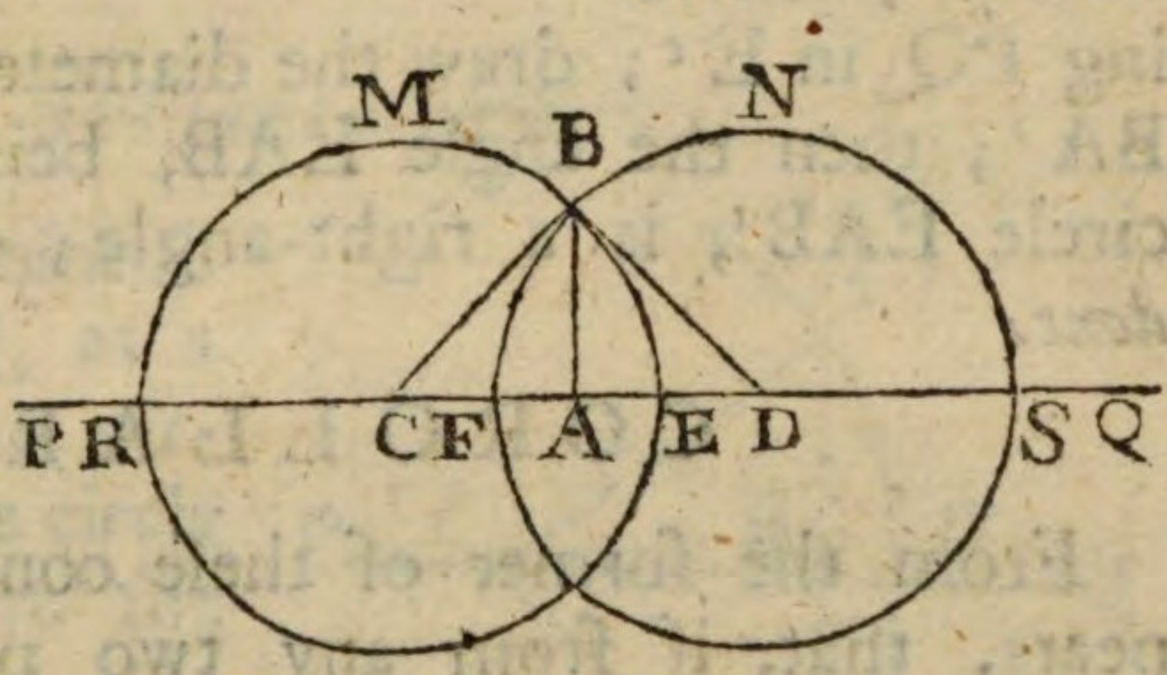


Draw the indefinite line AF[c]; from which take away AB = CD[d]; *and the thing is done.*

[c] Poſt. 1. and 2.
[d] 1. 5.

PROBLEM III.

At a given point (A) *in an infinite right-line* (PQ) *to erect a perpendicular.*



In the line propounded, take two equal diſtances AC, and AD[e]; and from the centers C[f] and D[g], with any equal radii greater than AC (or AD), let two circles EMR and FNS be deſcribed; which will cut each other: and, if from the point B of their interſection, you draw BA, *the thing is done.*

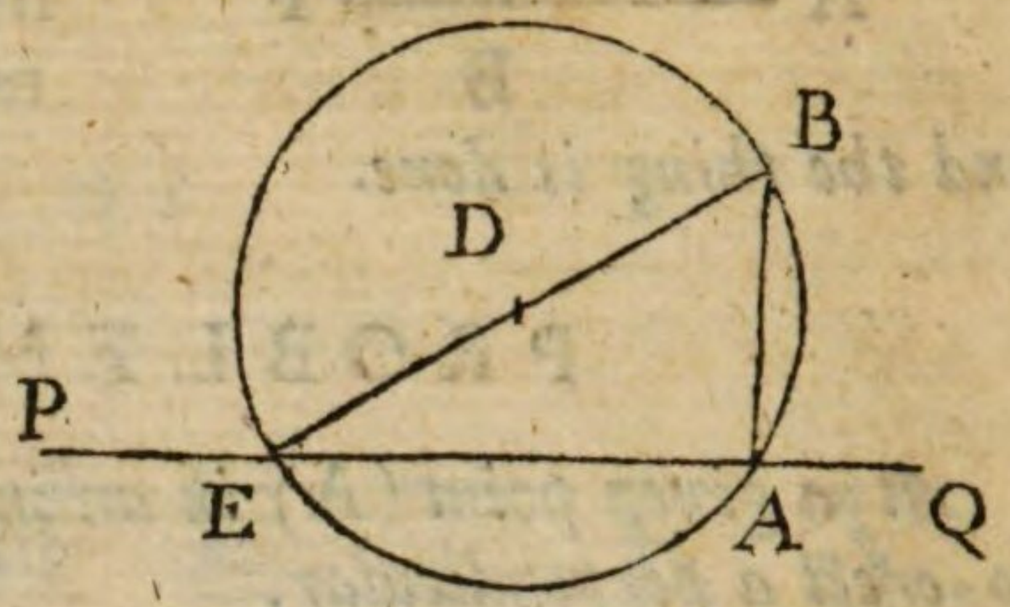
[e] 1. 5.
[f] Poſt. 3.
[g] Poſt. 1.

For, let the points R, E, and F, S, be thoſe wherein the infinite line PQ interſects the circumferences of the two circles EMR and FNS[h]: then AF being ⊐ FD[h] (or CR[i]) ⊐ AR[h]; and AS ⊏ DS[h] (or CE[i]) ⊏ AE[h], the point F falls within, and the point S without the circle EMR; and ſo the two circles cut each other[l]. If therefore, from the point of interſection, BC and BD be drawn; then the triangle CBD being iſoſceles[m], the angles BCD, BDC at the baſe thereof, will be equal[n]; and ſo, CA being = AD[k], and CB = BD[m], the angle CAB is alſo = DAB[p].

[h] Ax. 2.
[i] Hyp.
[l] Def. 11.
[m] 3. Conſtr. and Def. 33. of 1.
[n] *Obſ. on* Ax. 10.
[k] Conſtr.
[p] Ax. 10.

Otherwiſe.

Otherwiſe.



From any point D above the line PQ, as a center, thro' the given point A, let the circumference of a circle be deſcribed[q], interſecting PQ in E[r]; draw the diameter EDB, and alſo BA[s]; then the angle EAB, being in the ſemicircle EAB[t], is a right-angle; *which was to be done.*

[q] Poſt. 3.
[r] Cor. to 4. 3.
[s] Poſt. 1.
[t] 13. 3.

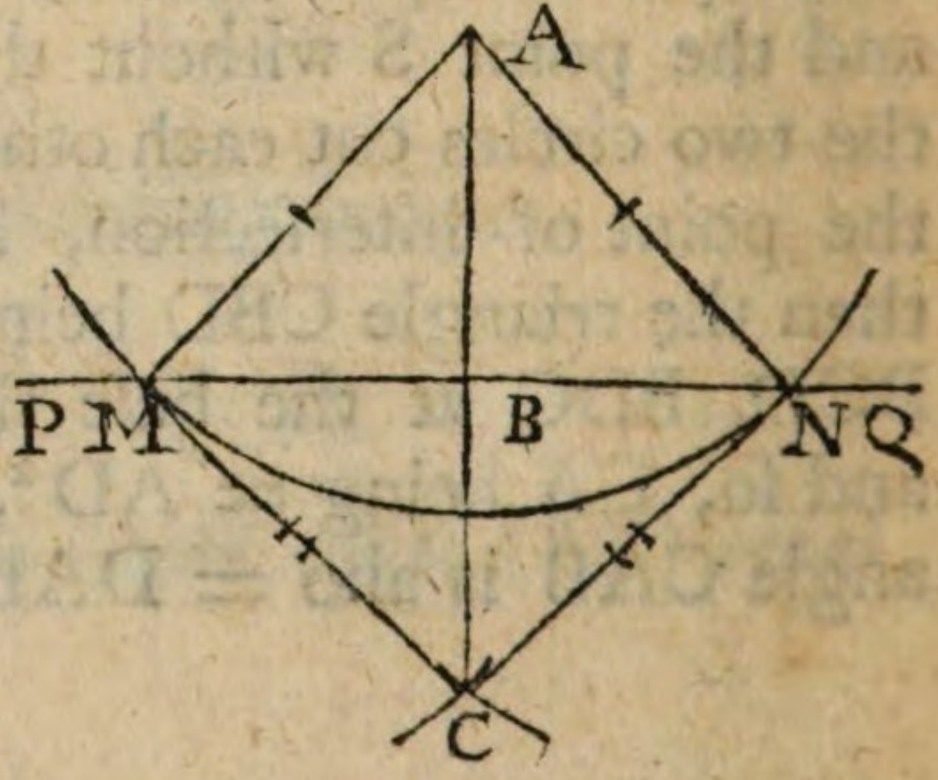
COROLLARY.

From the former of theſe conſtructions it appears, that, if from any two points, with two equal radii, greater, each, than half the diſtance of thoſe points, two circles be deſcribed; thoſe circles will cut each other.

PROBLEM IV.

From a given point (A) *upon an infinite right-line* (PQ) *to let fall a perpendicular* (AB).

From the given point A, as a center, let an arch of a circle be deſcribed, ſo as to paſs below PQ, and interſect it in M and N; from which points, with any equal radii, greater than half MN, let two other arches be alſo deſcribed, and from the point of



their

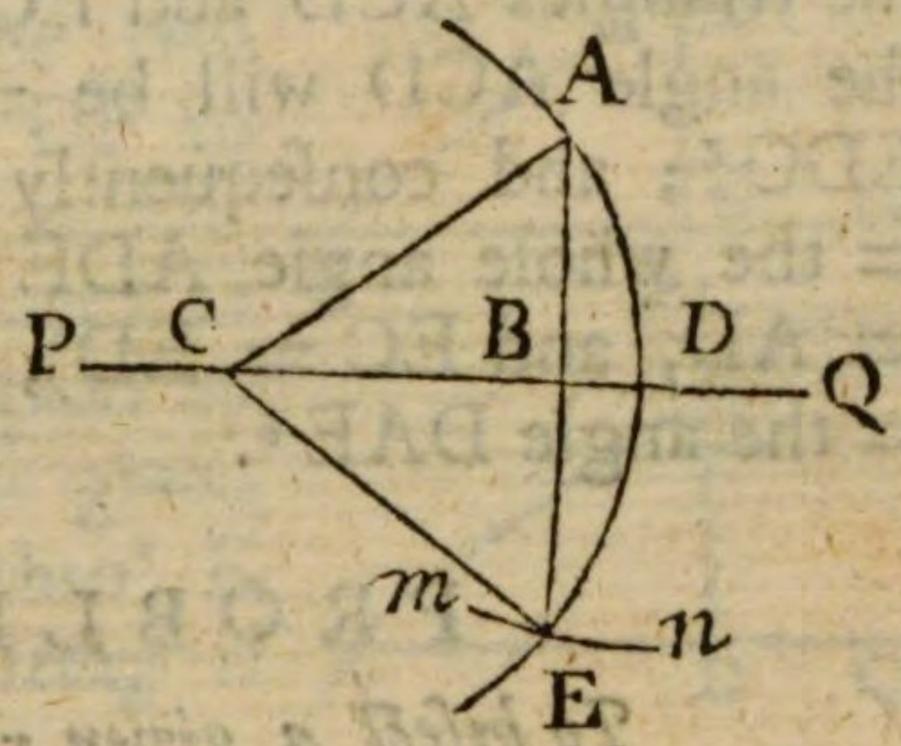
their interſection C, let the right-line CBA be drawn; which will be perpendicular to PQ.

For, let AM, AN, CM, and CN be drawn; then AM being = AN [u], and MC = NC [w], the angle AMB is = ANB [x], and CMB = CNB [x]; and conſequently AMC = ANC [y]: whence, (as AM = AN, and MC = NC) the triangles AMC, ANC are equal in all reſpects [z]; and ſo, the angle MAB being = NAB, the angle MBA is likewiſe = NBA [z].

[u] Def. 33. of 1.
[w] Hyp. and Def. 33. 1.
[x] *Obſ. on* Ax. 10.
[y] Ax. 4.
[z] Ax. 10.

The ſame otherwiſe.

From any point C in the line PQ, as a center, let the circumference of a circle be deſcribed thro' the given point A [a], interſecting PQ in D [b]; and from the center D, with a radius equal to the diſtance of the points A and D, let another circle *m*E*n* be alſo deſcribed, cutting the former ADE in E; then draw ABE for the perpendicular required.



[a] Poſt. 3.
[b] Conſtr.

For, conceiving right-lines to be drawn from C and D, to A and E, the triangles ACE, ADE will be both of them iſoſceles [b]; and ſo the demonſtration is the ſame with that of the preceding method.

PROBLEM V.

To biſect, or divide into two equal parts, any given right-lined angle (PAQ).

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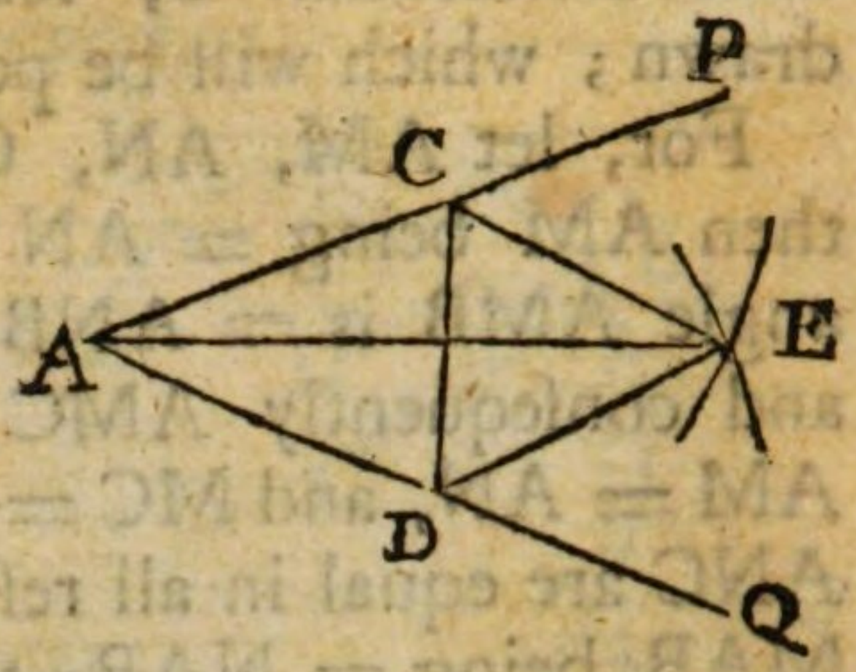


In the lines containing the given angle, take AC = AD [c]; and upon the centers C and D, with any equal radii, let two circles be described [d], so as to intersect each other; and from the point of intersection E draw EA, *and the thing is done.*

[c] 1. 5.

[d] Post. 3.

For, let CD, CE, and DE be drawn [e]; then, the triangles ACD and ECD being both isosceles [f], the angle ACD will be = ADC, and ECD = EDC [g]; and consequently the whole angle ACE = the whole angle ADE [h]; whence (AC being = AD, and EC = ED), the angle CAE is also = the angle DAE [g].

[e] Post. 1.

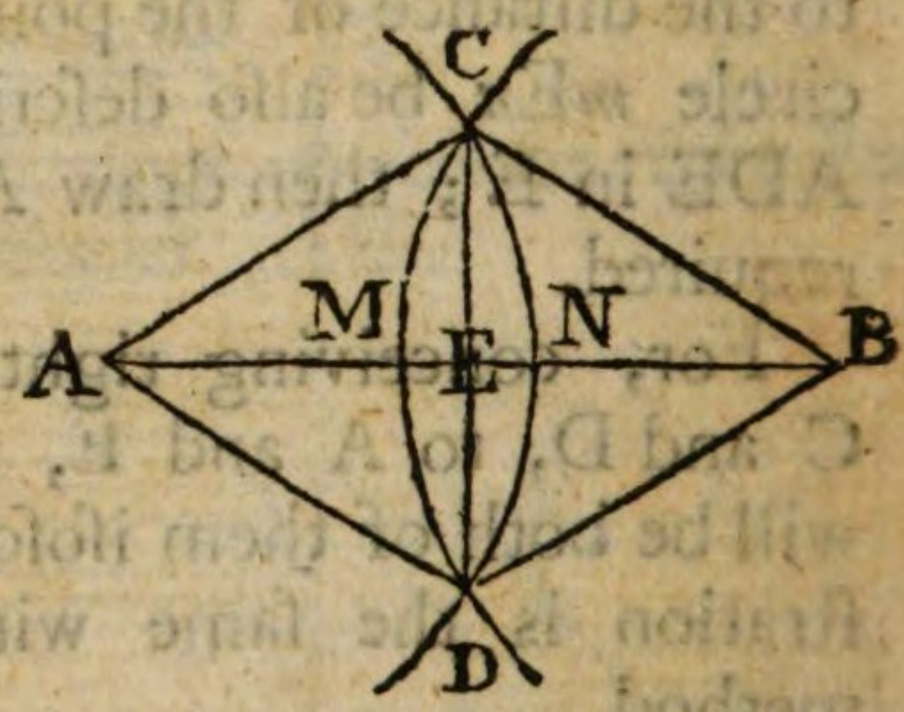
[f] Constr. and Def. 33. of 1.

[g] Ax. 10.

[h] Ax. 4.

PROBLEM VI.

To bisect a given right-line (AB).



From the extremes A, B, of the given line, with equal radii, describe two circles, so as to cut each other [i]; and between the two points of intersection draw CD, cutting AB in E; *and the thing is done.*

[i] Post. 3. and Cor. to 3. of 5.

For, if AC, AD, BC and BD be drawn, the triangles ACB, ADB being isosceles [k], thence is the angle CAB = CBA [l], and DAB = DBA [l]; and consequently CAD = CBD [m]: whence the triangles ACD and BCD are equal in all respects [n]; and so the angle ACE being = BCE, AC = BC, and CE common, thence is AE also = BE [n].

[k] Constr. and Def. 33. of 1.

[l] *Obs. on* Ax. 10.

[m] Ax. 4.

[n] Ax. 10.

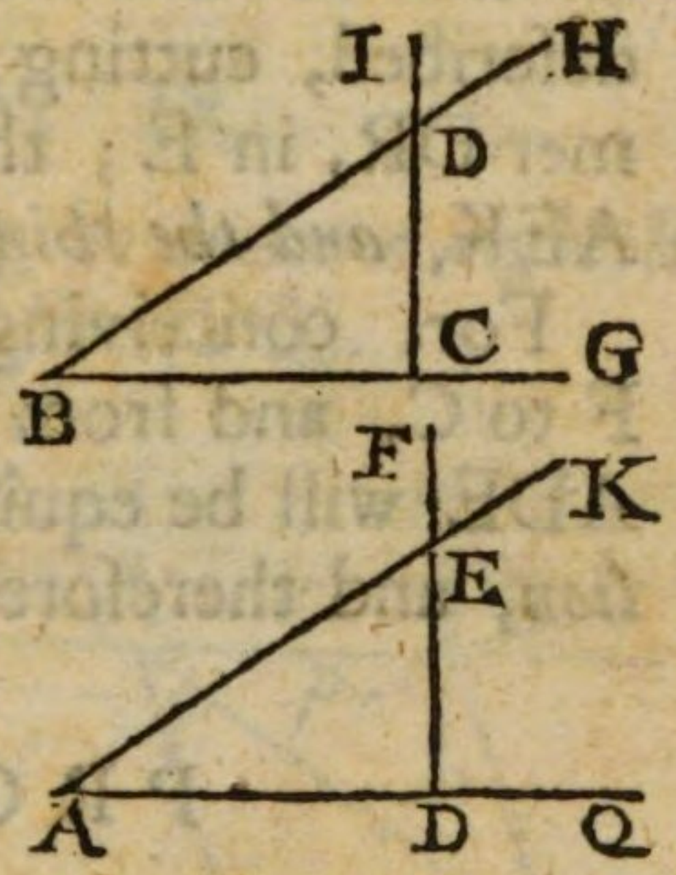
COROL-

COROLLARY.

Hence, it is manifeſt, that CD not only biſects AB, but is alſo perpendicular to it [n]. [n] Ax. 10. and Def. 8. of 1.

PROBLEM VII.

From a given point (A) *in a given right-line* (AQ), *to draw a line* (AK) *which ſhall make with the former an angle, equal to an angle given* (HBG).



In BG and AQ take two equal diſtances BC, AD°; and at C and D erect the two perpendiculars CI and DF to BG and AQ [p]; and in DF take DE equal to the part CD of the former, intercepted by the lines containing the given angle HBG [q]; then thro' E draw AEK [r], *and the thing is done* [s].

° 2. 5.

[p] 3. 5.

[q] 1. 5.

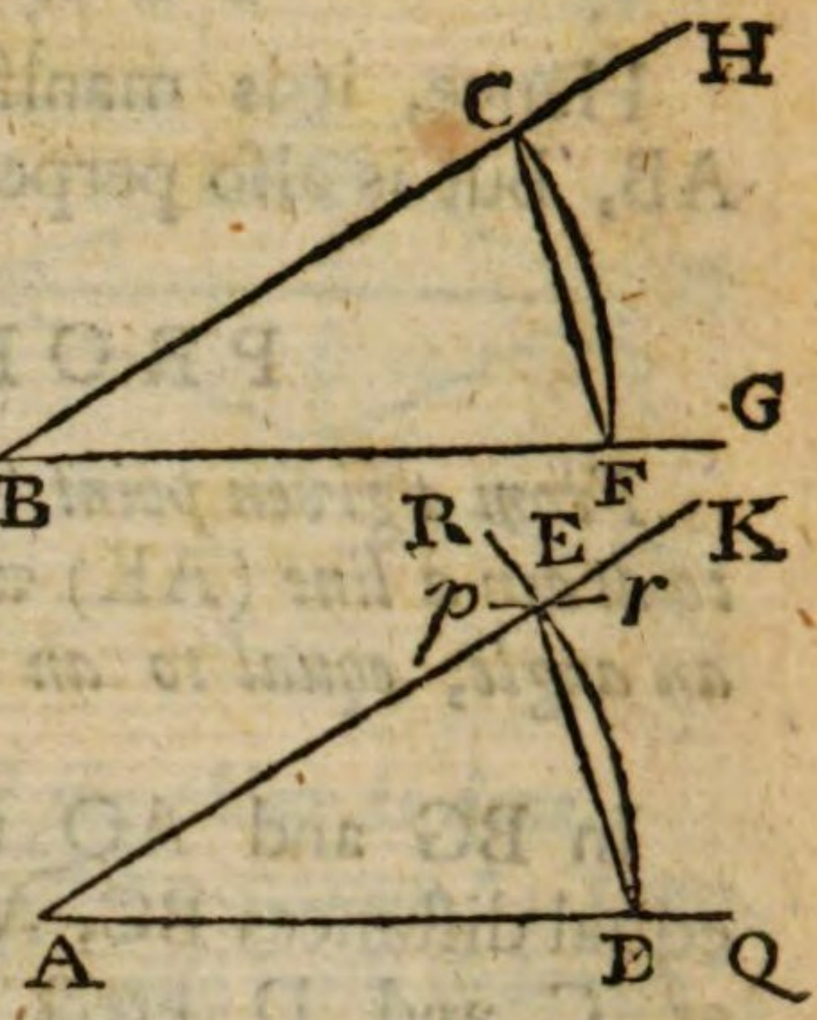
[r] Poſt. 1.

[s] Ax. 10.

SCHOLIUM.

Having, in the ſeven preceding problems, effected and demonſtrated, *by means of the axioms only*, whatever was aſſumed in the fourth poſtulate, *as barely poſſible*; we are now authoriſed, by the moſt rigid laws of geometrical reaſoning, to make uſe of any theorem or concluſion, whatſoever, derived in the preceding books, in virtue of thoſe aſſumptions, by which the proceſs and reſult can be rendered the moſt obvious and eligible.—Accordingly, by having recourſe here to Theorem XIV. of the firſt book, we ſhall be able to arrive at a conſtruction of the laſt problem, better adapted to practice than that above laid down.

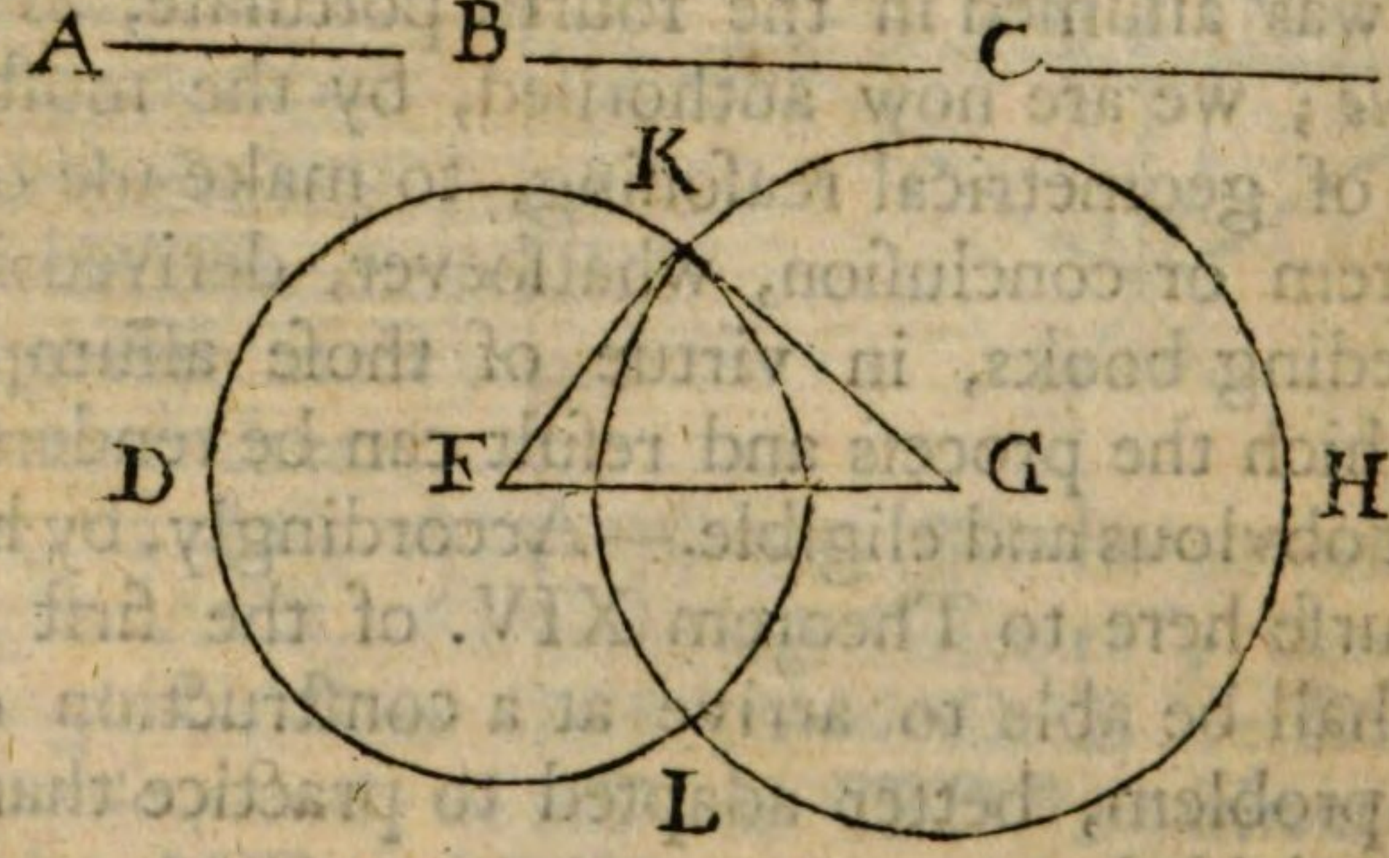
From the centers A and B, at any equal distances AD, BF, let two arcs of circles, FC, DR, be de-
* Post. 3. scribed *, intersecting the given lines in D, F, and C; also from D, with a radius equal to the distance of the points F, C, let another circular arch *p*E*r* be described, cutting the former DR, in E; then draw AEK, *and the thing is done.*



For, conceiving right-lines to be drawn from F to C, and from D to E, the triangles BFC and ADE will be equilateral to each other, *by construction*, and therefore equiangular also *.
* 14. 1.

PROBLEM VIII.

To describe a triangle, whose three sides shall be equal to three given lines (A, B, C); *provided any two of them, taken together, be greater than the third.*



Make FG = B *, and from the centers F and G, with the intervals, or distances A and C, let
* 2. 5.
two

two circles DKL, HKL be described [x]; which will cut each other; and, if from the point of intersection K, the lines KF and KG be drawn, then will FKG be the triangle required.

[x] Post. 3.

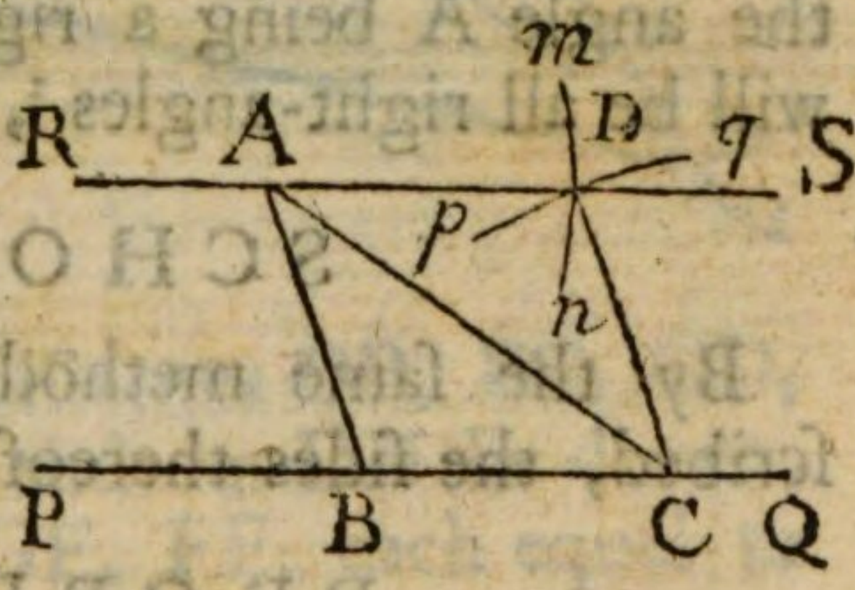
For, the distance FG of the two centers, is less than the sum of the radii A and C [y]; and greater than their difference (because B + A being ⊏ C [z], thence is B ⊏ C — A [y]); therefore the two circles cut each other [a]: consequently FK = A, FG = B, and GK = C [b].

[y] Ax. 6. 1.
[z] Hyp.
[a] 9. 3.
[b] Def. 33. of 1.

PROBLEM IX.

Through a given point (A), *to draw a right-line* (RS) *parallel to a given right-line* (PQ).

At any point B in the given line, make BC equal to the distance of the points A and B [c]; and from the centers A and C, with an interval equal to CB, let two circles be described [d], then thro' their intersection D, let the line RS be drawn, *and the thing is done.*



[c] 2. 5.
[d] Post. 3.

Let AB, and AC be drawn [e]. It is plain that the two circles will cut each other [f], because the sum of their semi-diameters (= AB + BC [g]) is greater than AC [h]: therefore, if ADS and CD be also drawn, then will AB = BC = CD = DA [i], and therefore RS parallel to PQ [k].

[e] Post. 1.
[f] 9. 3.
[g] Constr.
[h] 19. 1.
[i] Ax. 1.
[k] 25. 1.

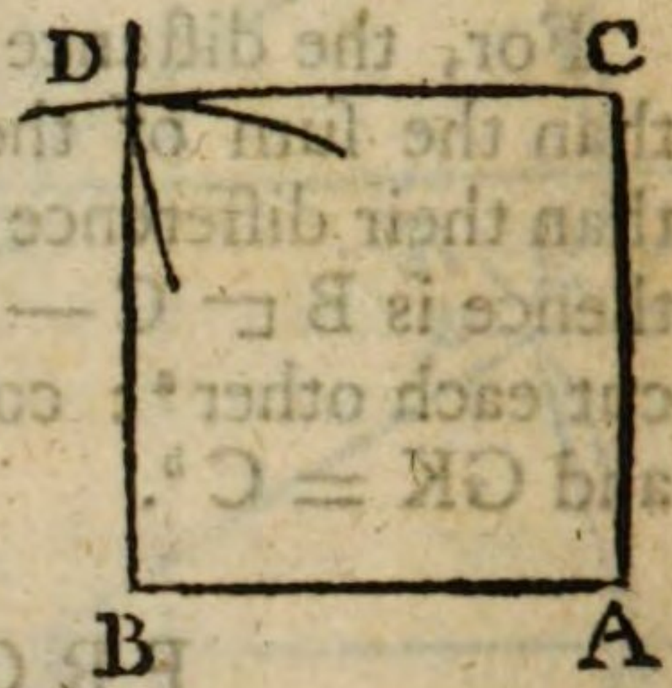
The same otherwise.

From A, to any point in PQ, draw AB [l]; make the angle SAB = PBA [m]; and then AS will be parallel to PQ [n].

[l] Post. 1.
[m] 7. 5.
[n] 8. 1.

PROBLEM X.

Upon a given line (AB) *to describe a square* (ABCD).



Make AC perpendicular, and equal, to AB [o]; and from the centers B and C, let two circles, with the radius AB or AC, be described [p], intersecting each other [q] in D; from which point draw DB and DC, *and the thing is done.*

[o] 3. 5. and 1. 5.
[p] Post. 3.
[q] 9. 3.

For, all the four sides being equal, by construction, the figure is a parallelogram [r]; and therefore, the angle A being a right-angle [s], the other three will be all right-angles [t], and ACDB a square [v].

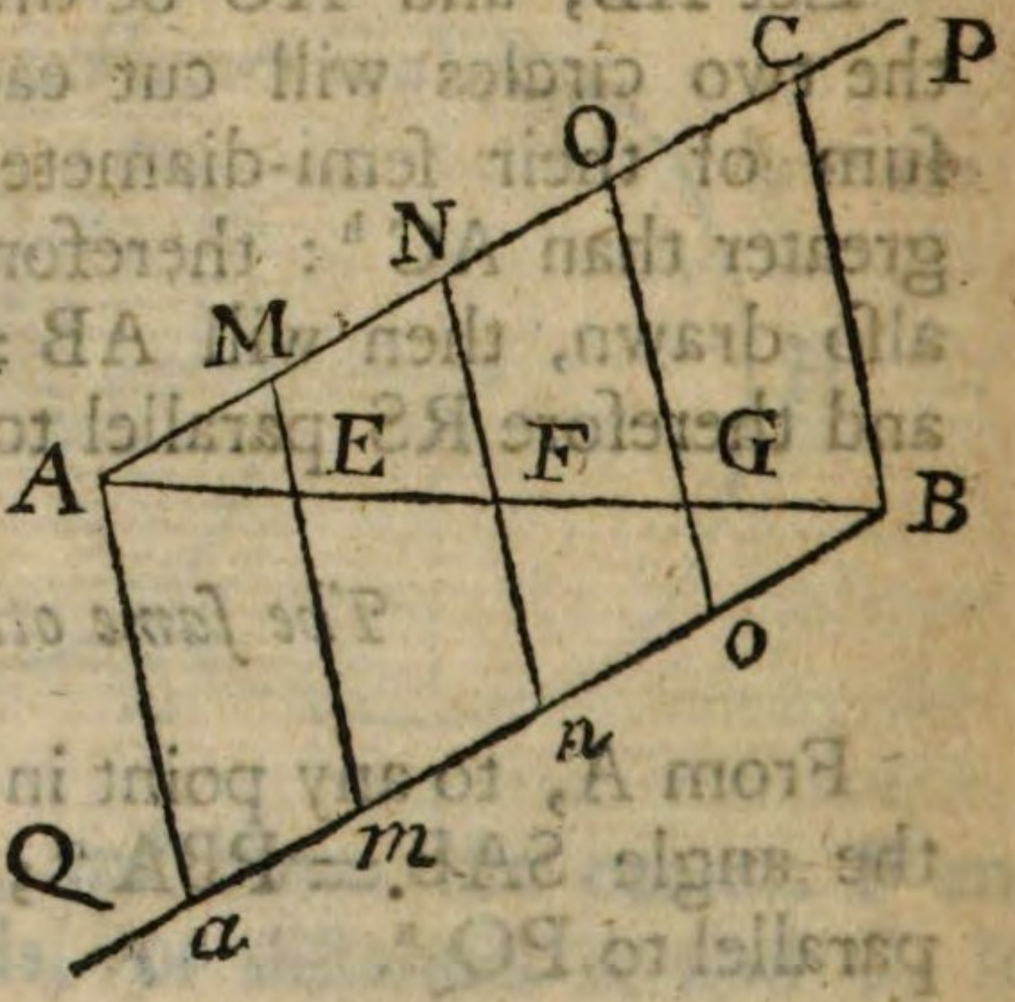
[r] 25. 1.
[s] Constr.
[t] Cor. to 24. 1.
[v] Def. 26.

SCHOLIUM.

By the same method a rectangle may be described, the sides thereof being given.

PROBLEM XI.

To divide a given line (AB) *into any proposed number of equal parts.*



From the extremes of the given line AB, draw two indefinite lines AP, BQ parallel to each other [u]; in each of which lines let there be taken as many equal distances AM, MN, NO, OC; B*o*, *on*, *nm*, *ma*, (of any

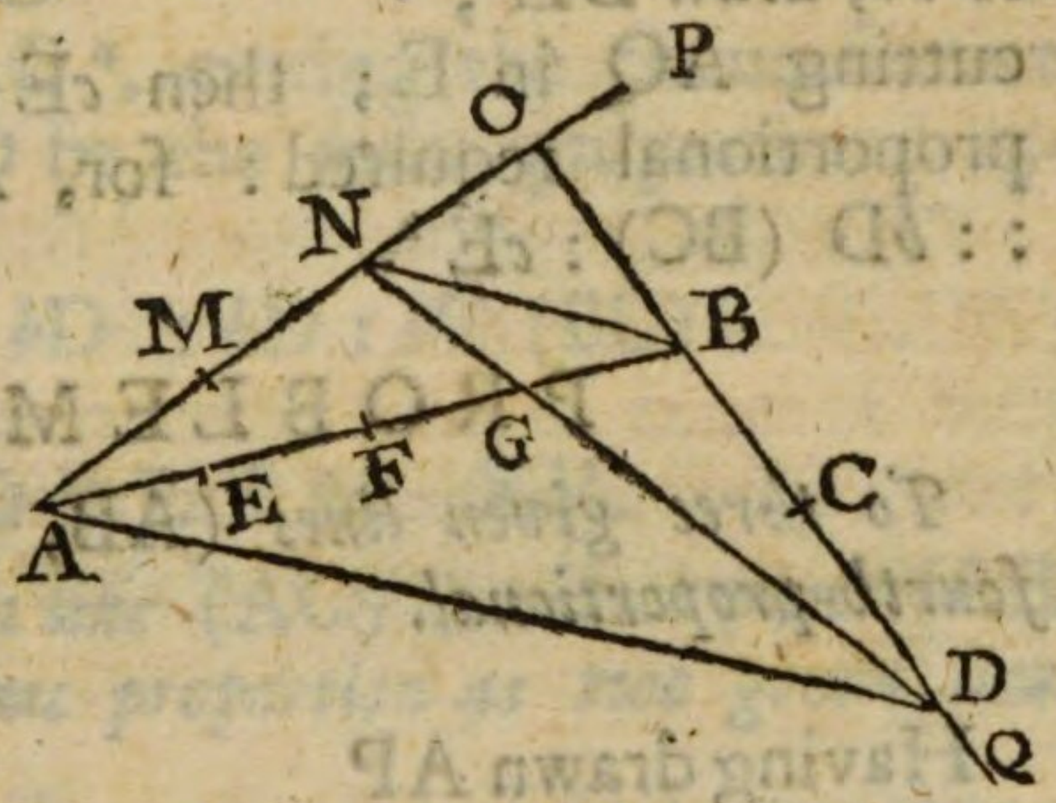
[u] 9. 5.

length

length at pleaſure) as you would have AB divided into [w]; then draw M*m*, N*n*, O*o*, interſecting AB [w] in E, F, G, *and the thing is done.* [w] 1. 5.

For, MN and *mn* being equal and parallel [x], FN will be parallel to EM [y]; and in the ſame manner will GO be parallel to FN: therefore, AM, MN, NO, *&c.* being all equal [x], AE, EF, FG, will likewiſe be equal [z]. [x] Conſtr. [y] 26. 1. [z] Cor. 1. to 27. 1.

The ſame otherwiſe.



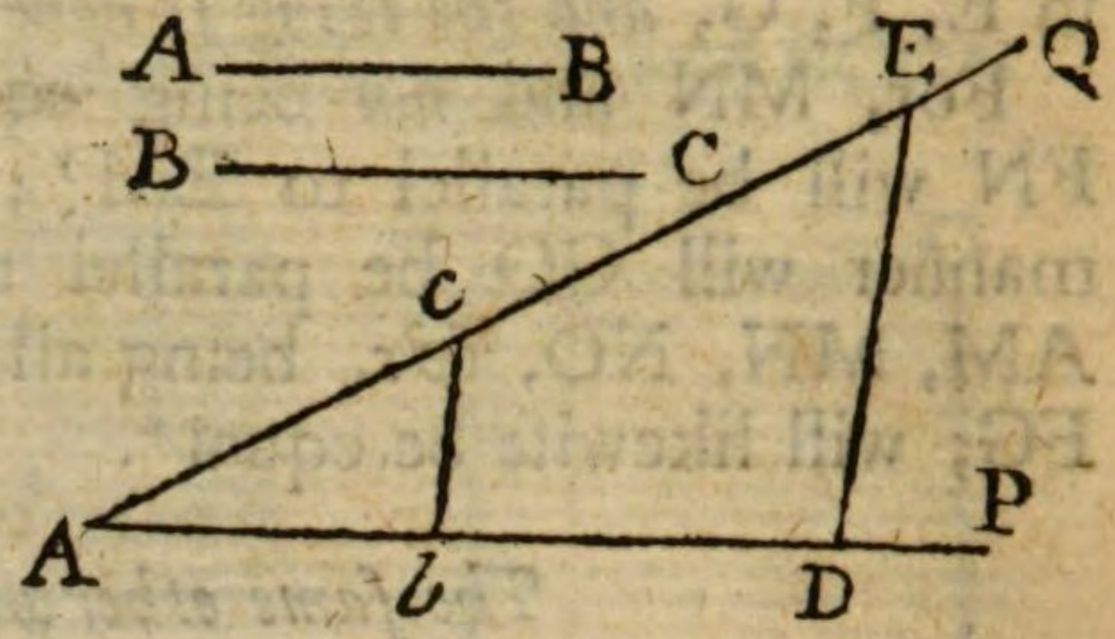
In any right-line AP drawn from A, take as many equal diſtances (AM, MN, NO) wanting one, as you would have AB divided into; then, having drawn the indefinite line OBQ, in it take an equal number of parts or diſtances OB, BC, CD, each of the length of OB, and let DN be drawn, cutting AB in G; make GF, FE, each equal to BG, *and the thing is done.*

For, if AD and BN be drawn, they will be parallel [a] (becauſe OA : ON :: OD : OB [b]); and ſo, the triangles BNG, ADG, being equiangular [c], it will be BG : AG :: BN : AD [d] :: ON : OA [d]. Therefore BG is the ſame part of AG, as ON is of OA. [a] Cor. to 12. 4. [b] Conſtr. [c] 3. and 7. 1. [d] 14. 4.

PROBLEM XII.

To two given lines (AB, BC) *to find a third-proportional.*

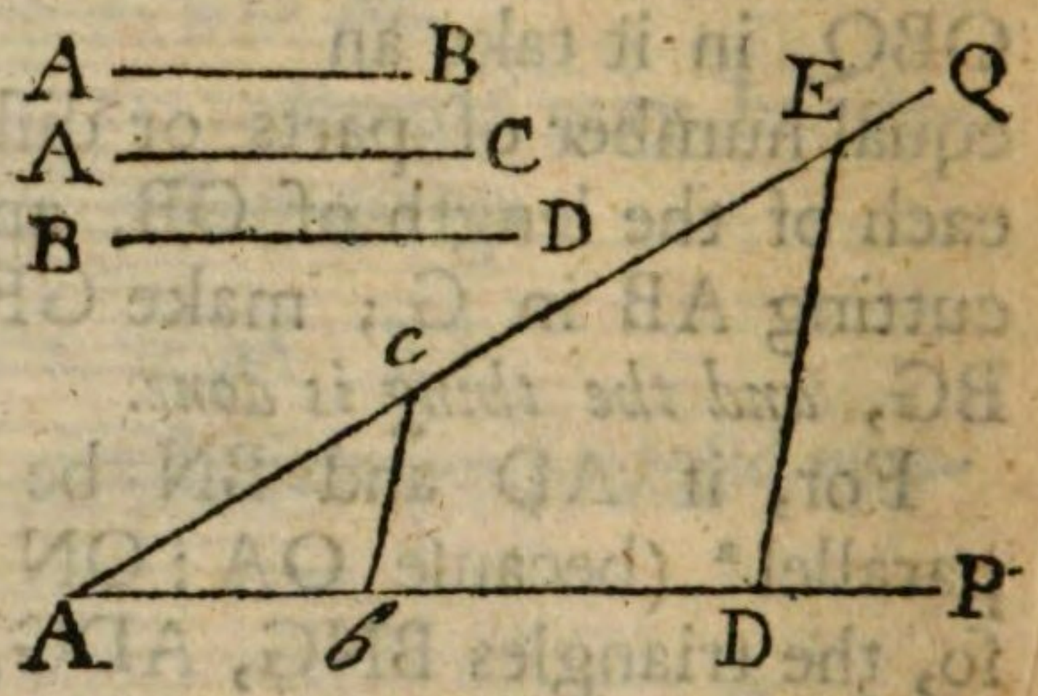
From any point A, draw two indefinite lines AP, AQ, in which take $Ab = AB$, $Ac = BC$, and
[e] 1. 5. $bD = BC$ [e]; draw bc, and parallel
[f] 9. 5. to bc, draw DE [f], cutting AQ in E; then cE will be the third-proportional required: for, Ab (AB) : Ac (BC)
[g] 12. 4. : : bD (BC) : cE [g].



PROBLEM XIII.

To three given lines (AB, AC, BD) *to find a fourth-proportional.*

Having drawn AP and AQ, as in the preceding problem, take therein $Ab = AB$, $Ac = AC$, and
[i] 1. 5. $bD = BD$ [i]; draw bc, and parallel to
[k] 9. 5. it, draw DE [k], intersecting AQ in E; then is cE *the fourth proportional required.*
[m] 12. 4. For, Ab (AB) : Ac (AC) : : bD (BD) : cE [m].

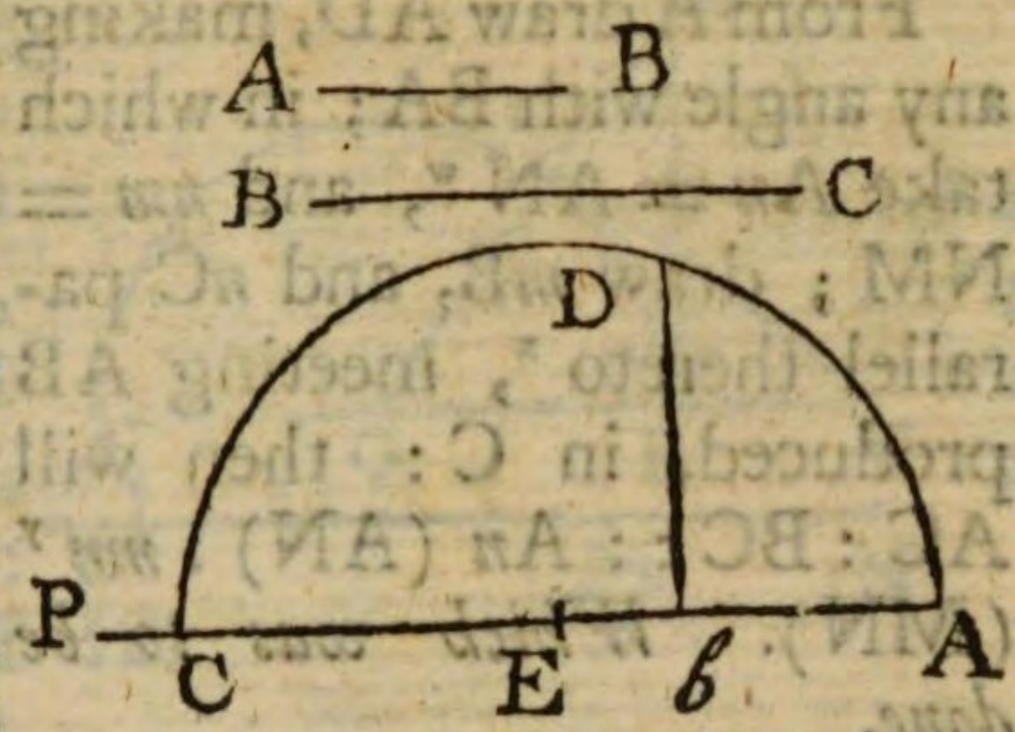


PROBLEM XIV.

Between two given lines (AB, BC) *to find a mean-proportional.*

In

In the indefinite line AP, take A*b* = AB, and *b*C = BC [a]; bisect AC in E [b], and from the center E, at the distance of EA, or EC, let a semicircle ADC be described [p]; erect *b*D perpendicular to AC [q], cutting the circumference in D; then will *b*D be the mean-proportional required.



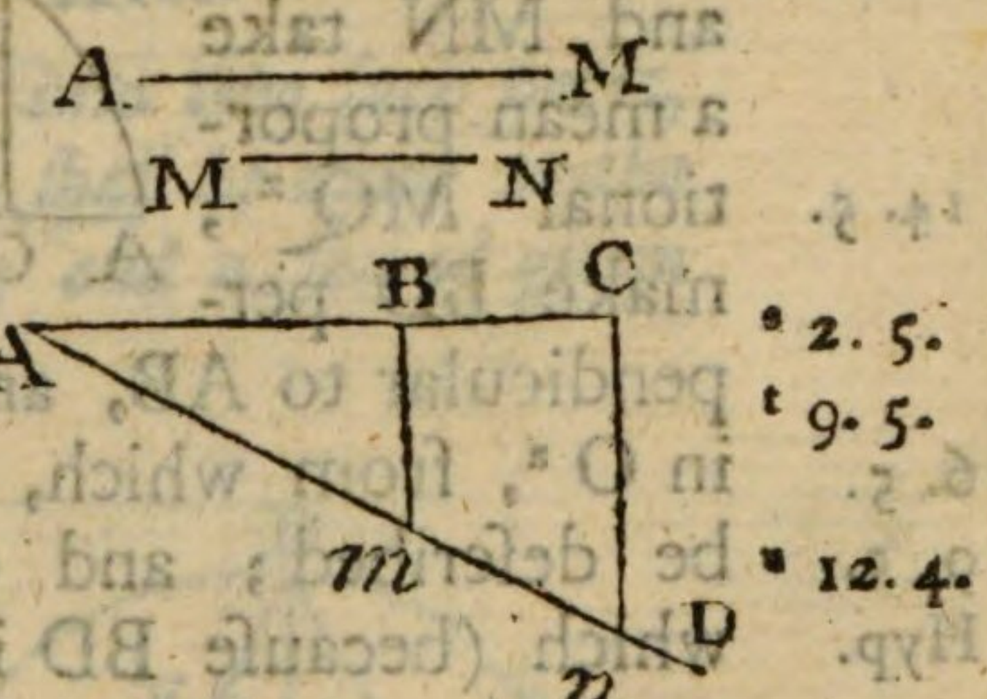
[a] 1. 5.
[b] 6. 5.
[p] Post. 3.
[q] 3. 5.

For, A*b* (AB) : *b*D :: *b*D : *b*C (BC) [r]. [r] 19. 4. and it's Corol.

PROBLEM XV.

To divide a given line (AC) *into two parts* (AB, BC) *having the same proportion as two given lines* (AM, AN).

From A draw AD, making any angle with AB; in which take A*m* = AM, and *mn* = MN [s]; draw *n*C, and *m*B parallel thereto [t], meeting AC in B. Then will AB : BC :: A*m* (AM) : *mn* [u] (MN). *Which was to be done.*



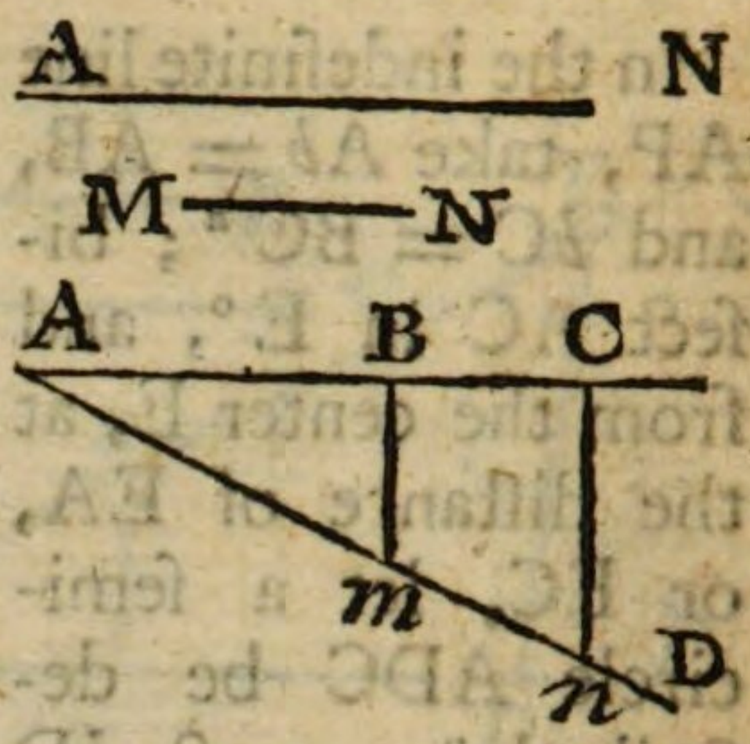
[s] 2. 5.
[t] 9. 5.
[u] 12. 4.

PROBLEM XVI.

To add a line (BC) *to a line given* (AB), *so that the whole compounded line* (AC) *shall be in proportion to the part added, as one given line* (AN) *is to another* (MN).

From

From A draw AD, making any angle with BA; in which take $An = AN$ [w], and $nm = NM$; draw mB, and nC parallel thereto [x], meeting AB produced, in C: then will $AC : BC :: An\ (AN) : mn\ (MN)$ [y]. *Which was to be done.*

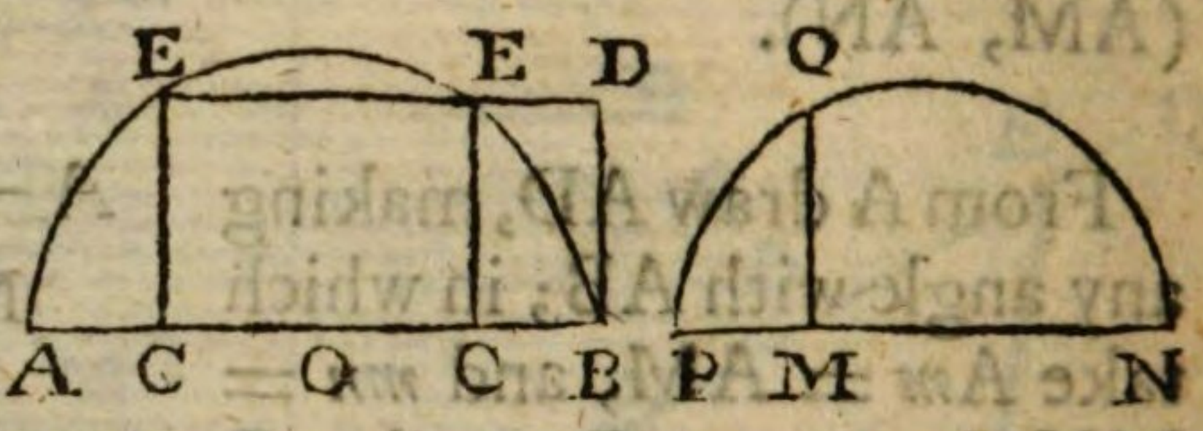


[w] 2. 5.
[x] 9. 5.
[y] 12. 4.

PROBLEM XVII.

To divide a given line (AB) *into two such parts* (AC, BC) *that the rectangle contained under them, shall be equal to the rectangle under two given lines* (PM, MN); *provided that the given rectangle is not greater than the square of half the line* (AB) *to be divided.*

Between PM and MN take a mean-proportional MQ [z]; make BD perpendicular to AB, and equal to MQ; bisect AB in O [a], from which, as a center, let a semi-circle be described; and draw DE parallel to BA [b], which (because BD is less than the radius [c]) will meet the circle in some point E; from which, upon AB let fall the perpendicular EC: so shall $AC \times BC = {}^{d}EC^2 = DB^2\ (= QM^2) = PM \times MN$ [e]. *Which was to be done.*



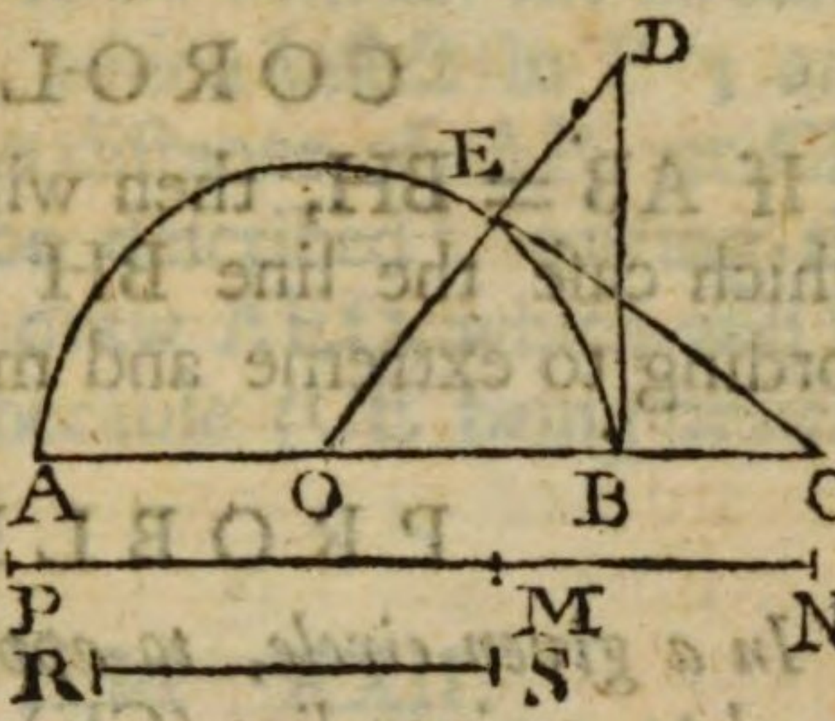
[z] 14. 5.
[a] 6. 5.
[b] 9. 5.
[c] Hyp.
[d] Cor. to 19. 4.
[e] Hyp. and 10. 4.

PROBLEM XVIII.

To a given line (AB) *to add another line* (BC), *such, that the rectangle under the whole compounded line* (AC) *and the part added, shall be equal to a rectangle under two given lines* (PM, MN).

Between

Between PM and MN take a mean-proportional RS [f]; make BD perpendicular to AB [g], and equal to RS [h]; biſect AB in O [i], draw OD, and take OC = OD [k]: ſo ſhall AC × BC = PM × MN, *as was to be done.*



[f] 14. 5.
[g] 3. 5.
[h] 2. 5.
[i] 6. 5.

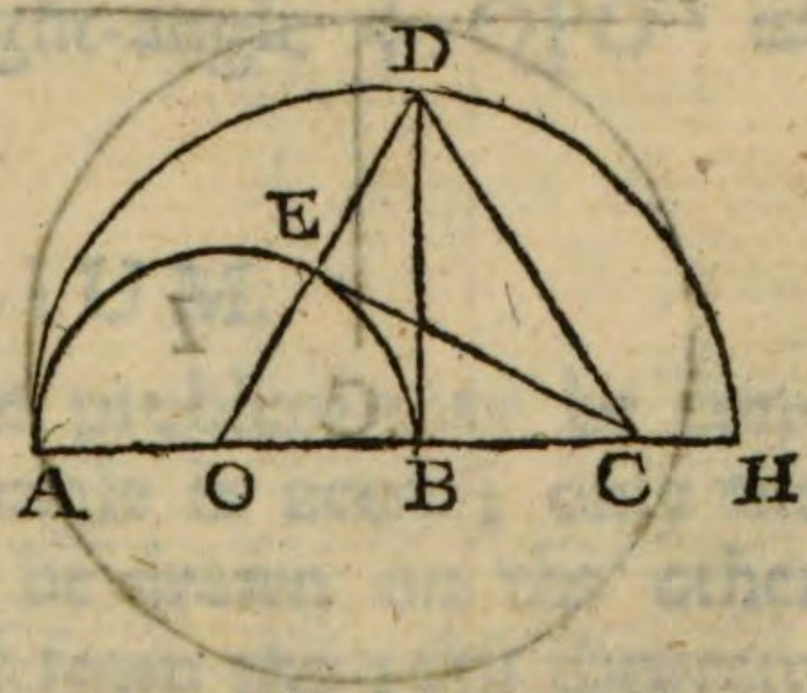
For, if thro' A and B, from the center O, the circumference of a circle be deſcribed, cutting DO in E, and E, C be joined; then, the triangles OCE, ODB (having OE = OB, OC = OD [k], and the angle EOB common) will be equal in all reſpects [l]; and ſo, EC being a [m] tangent to the circle in E, we have [n] AC × BC = CE² = [o] BD² = RS² = PM × MN [p].

[k] Conſtr.
[l] Ax. 10.
[m] 6. 3.
[n] Cor. to 22. 3.
[o] 1. 2.
[p] 10. 4.

PROBLEM XIX.

To divide a given line (BH) *into two ſuch parts, that the ſquare of the one* (BC) *ſhall be equal to the rectangle under the other* (CH) *and a ſecond given line* (AB).

Taking BA in the ſame ſtrait line with BH, between them let a mean-proportional BD be found [q]; biſect AB in O [r]; draw OD, and make OC = OD; ſo ſhall BC² = CH × AB, *as was to be done.*



[q] 14. 5.
[r] 6. 5.

For, by the demonſtration of the precedent, AC × BC (= CE² = BD²) = AB × BH; from each of which taking away AB × BC, there remains BC² = AB × CH [s].

[s] 5. 2. and Ax 5. 1.

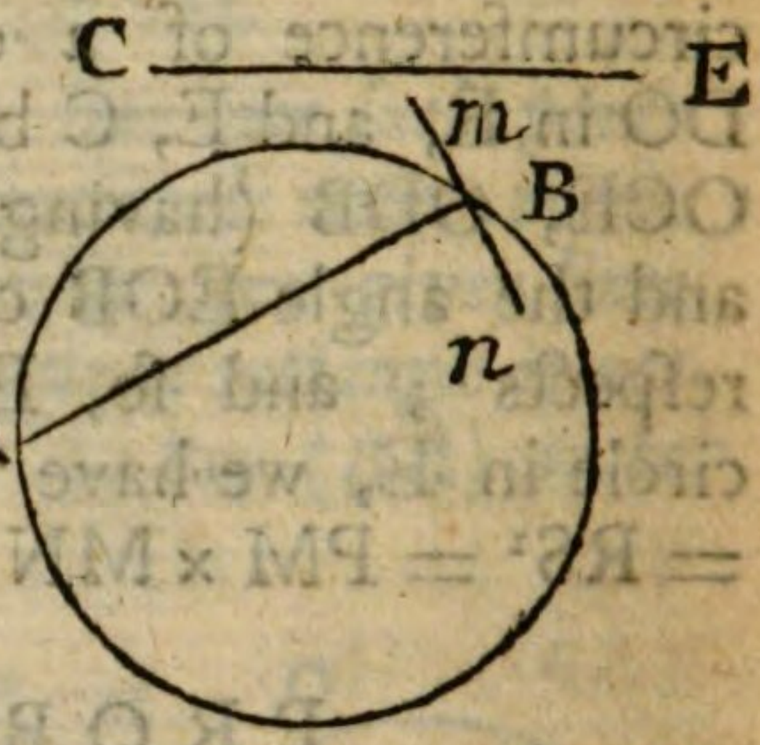
COROL-

COROLLARY.

If AB = BH, then will $BC^2 = BH \times CH$; in which case the line BH is said to be divided according to extreme and mean-proportion.

PROBLEM XX.

In a given circle, to apply, or inscribe a line (AB) *equal to a given line* (CE), *less than the diameter of the circle.*

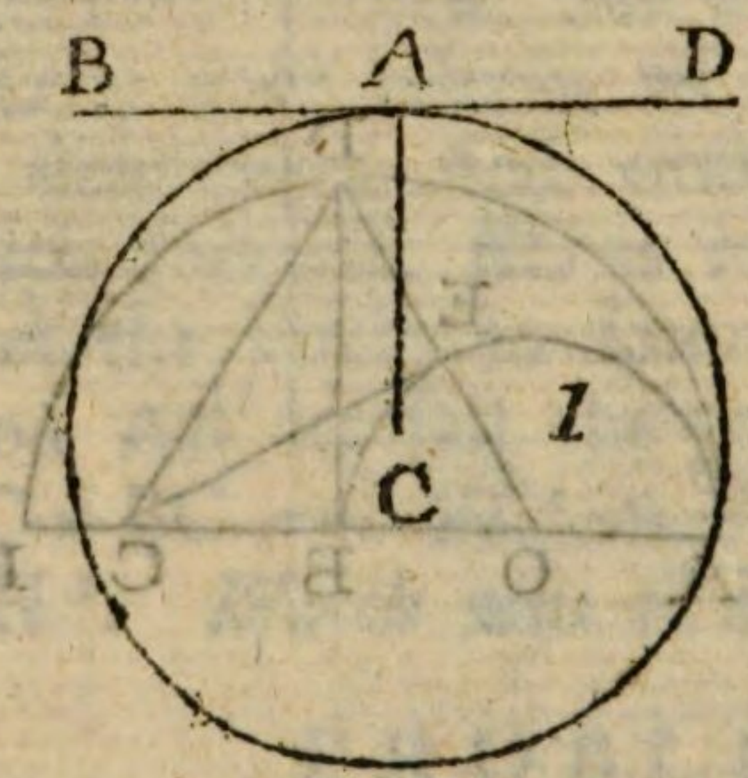


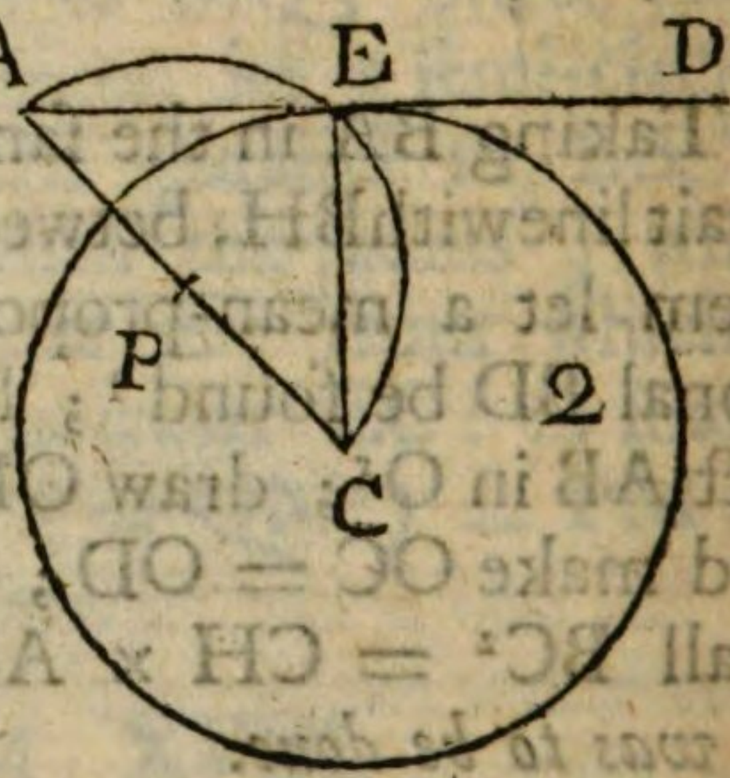
From any point A in the circumference, with the radius CE, let a circular arch *mn* be described [t], cutting the given circle in B [u]; then draw AB, *and the thing is done* [w].

[t] Post. 3.
[u] 9. 3.
[w] Def. 33. of 1.

PROBLEM XXI.

To draw a tangent to a given circle (C) *thro' a given point* (A).





Case I. If the given point be in the circumference; then, to the center C, draw AC, and perpendicular to AC [y] draw BAD; which will touch the circle at A [z].

[y] 3. 5.
[z] 6. 3.

Case II.

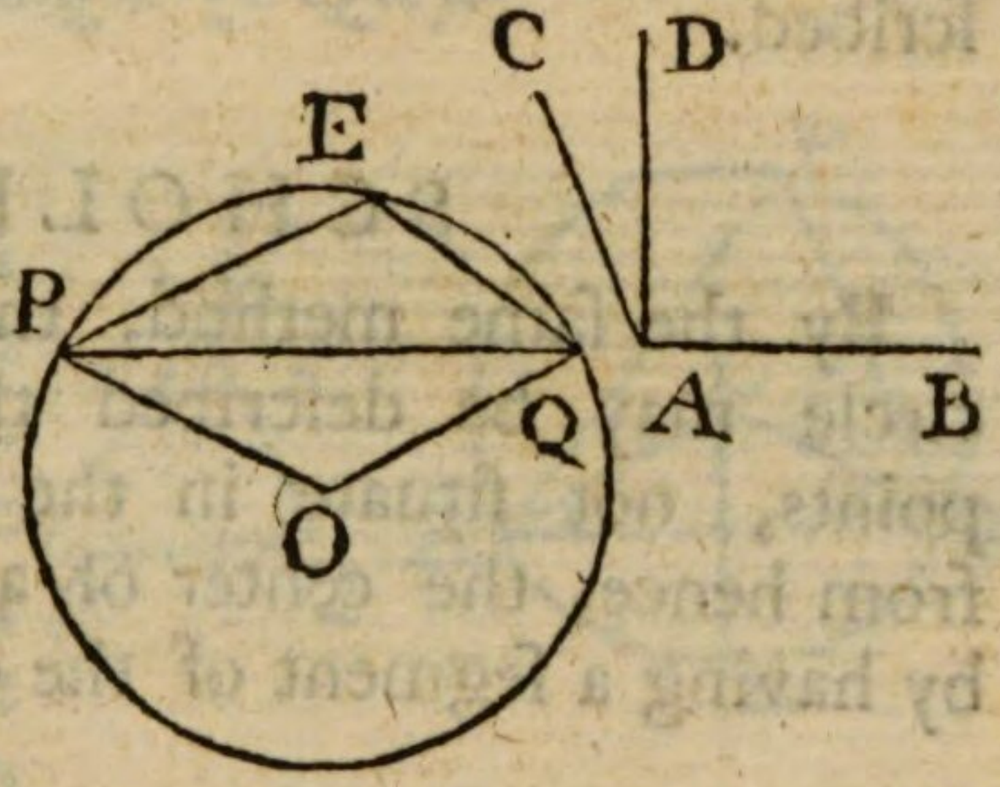
CASE II. If the point A be without the circumference; then draw AC, which biſect in P [b]; and from the center P, at the diſtance of AP, or CP, let a ſemi-circle AEC be deſcribed [c], cutting the given circle in E [d]; then draw AED, which will be the tangent required [e]; becauſe (CE being drawn) AEC is a right-angle [f].

[b] 6. 5.
[c] Poſt. 3.
[d] 9. 3.
[e] 6. 3.
[f] 13. 3.

PROBLEM XXII.

Upon a given line (PQ) *to deſcribe a ſegment of a circle* (PEQ) *to contain an angle* (E) *equal to a given angle* (BAC).

Make AD perpendicular to AB [g]; alſo make PQO, and QPO, each, equal to DAC [h] (the difference between the given angle and a right one); then upon the point of interſection O, as a center, at the diſtance of OP (or OQ), let a circle be deſcribed; *and the thing is done.*



[g] 3. 5.
[h] 7. 5.

For the angle E = right-angle + QPO [i] = DAB + DAC [k] = BAC.

[i] 16. 3.
[k] Conſtr.

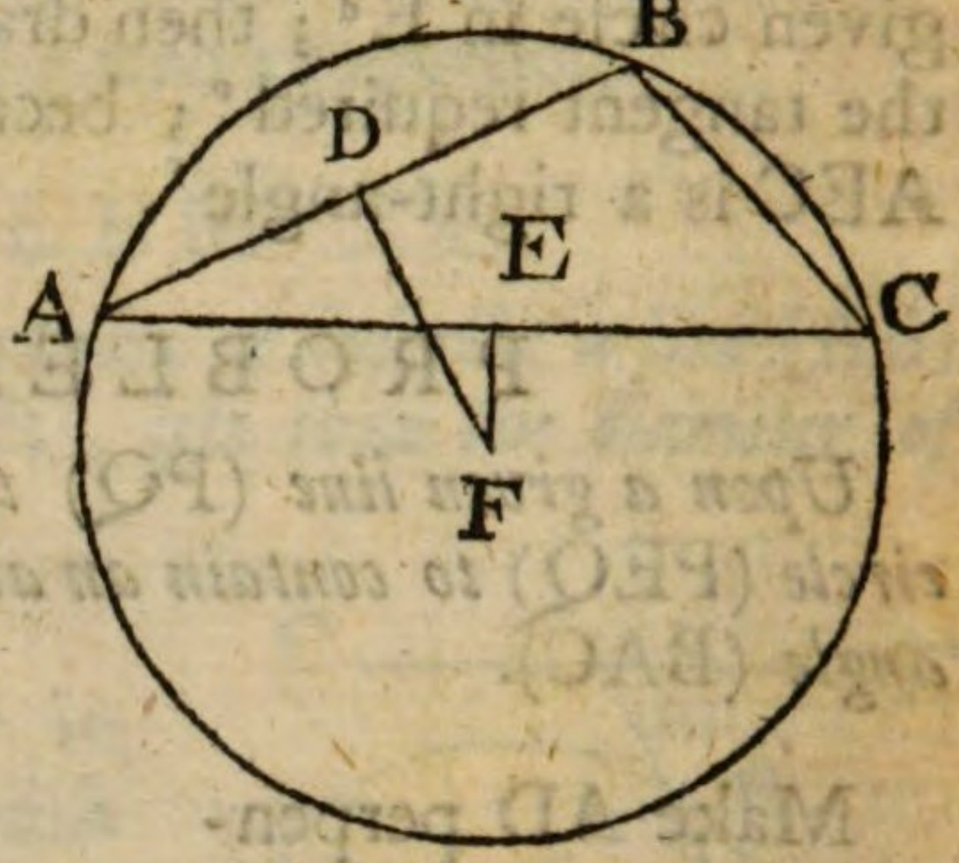
SCHOLIUM.

In the ſame manner the problem may be conſtructed, when the given angle is acute; only the lines PO, QO muſt then be drawn on the other ſide of (PQ) as is manifeſt from the 16th theorem of the 3d book.

PRO-

PROBLEM XXIII.

About a given triangle (ABC) *to describe a circle.*



Let any two sides, AB and AC, be bisected by two perpendiculars DF and EF [k]; which will intersect each other in the center (F) of the required circle [l]; from whence the circle may be described.

[k] 6. 5.

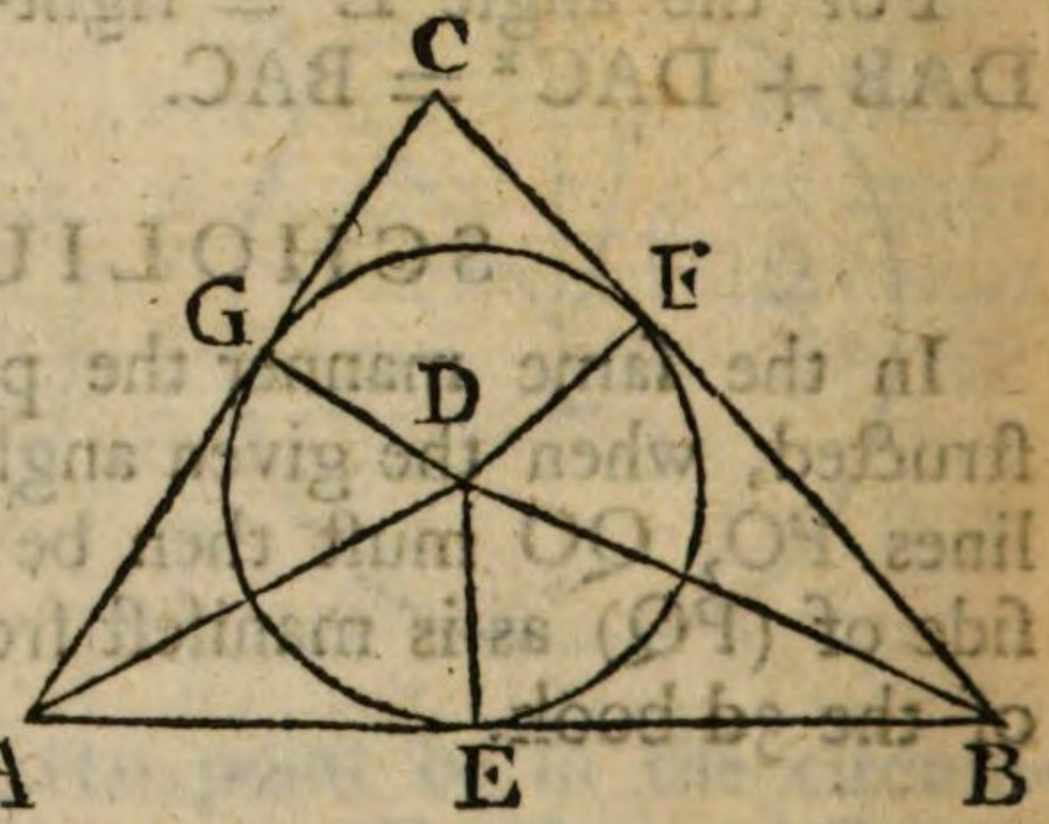
[l] Proof of 19. 3.

SCHOLIUM.

By the same method, the circumference of a circle may be described thro' any three given points, not situate in the same right-line: also from hence, the center of a circle may be found, by having a segment of the circle given.

PROBLEM XXIV.

To inscribe a circle in a given triangle (ABC).



Bisect any two of the angles, A and B, by the lines AD and BD [m], meeting each other in D; make DE perpendicular to AB [o]; then, if from the center D, at the distance of DE, a circle be described, it will touch all the sides of the triangle.

[m] 5. 5.

[o] 4. 5.

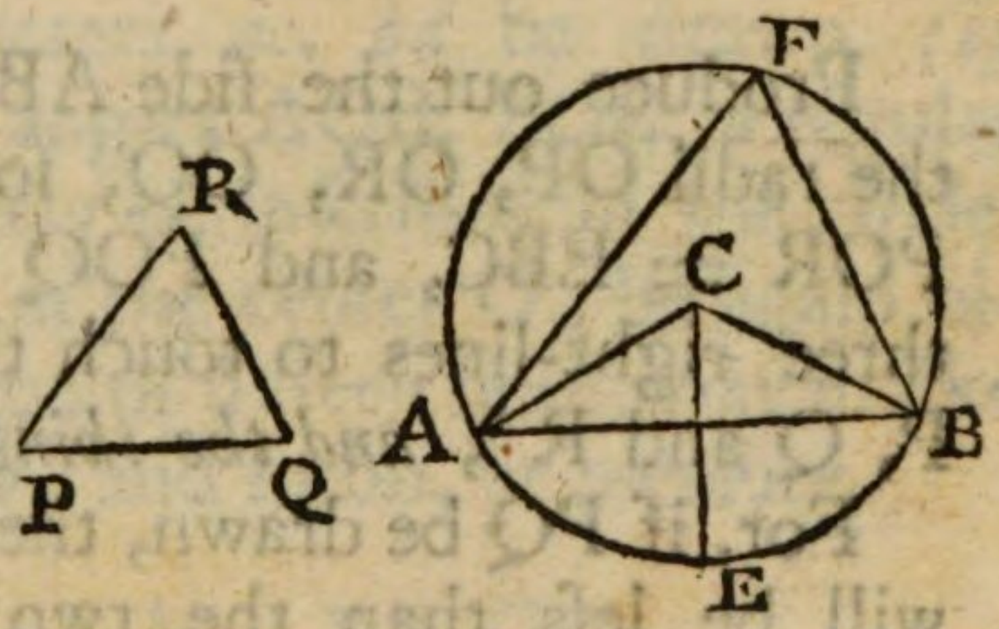
For,

For, let DG and DF be perpendicular to AC and BC [p]; then the triangles ADE, ADG, having two angles equal, each to each (by conſtruction) and AD common, will not only be equiangular [q], but alſo have DE = DG [r]. By the ſame argument DE = DF; therefore the circumference of the circle alſo paſſes through G and F [s]; but it touches the ſides of the triangle in thoſe points [t], becauſe G and F are right-angles [u].

[p] 4. 5.
[q] Cor. 1. to 10. 1.
[r] 15. 1.
[s] Def. 33. of 1.
[t] 6. 3.
[u] Conſtr.

PROBLEM XXV.

In a given circle (AFB) *to deſcribe a triangle, equiangular to a given triangle* (PQR).



From the center C, draw the radii CA, CE, CB, making the angles ACE and BCE equal, each, to the angle R [w]; join A, B, and make the angle ABF = Q [w], and from the point F, where BF cuts the circle, draw FA; ſo ſhall AFB be the triangle required.

[w] 7. 5.

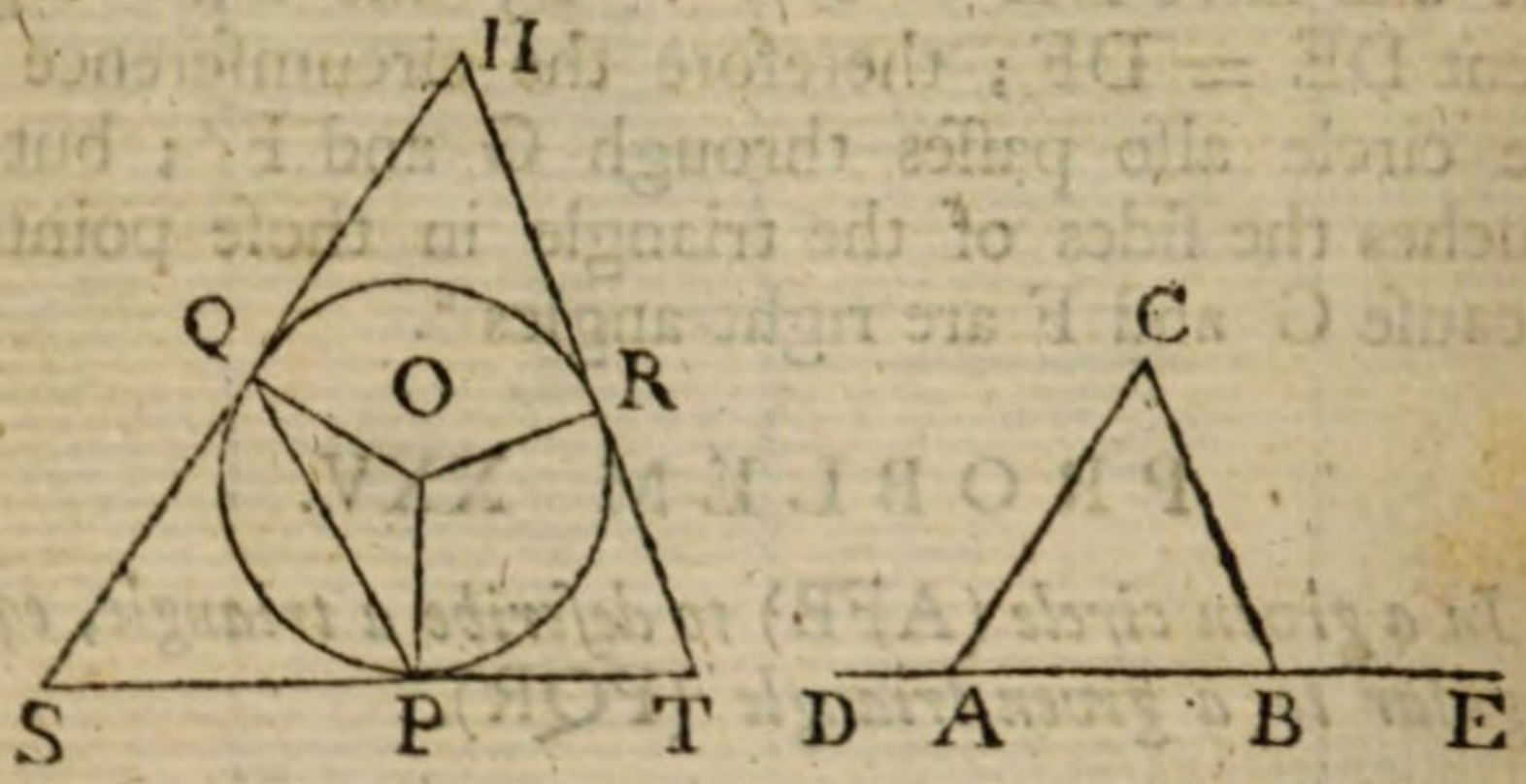
For, ABF = Q [x], F (= ACE [y]) = R [x]; and conſequently BAF = P [z].

[x] Conſtr.
[y] 10. 3. and Conſtr.
[z] Cor. 1. to 10. 1.

PRO-

PROBLEM XXVI.

About a given circle (O), *to describe a triangle, equiangular to a given triangle* (ABC).



Produce out the side AB both ways; and draw the radii OP, OR, OQ, so as to make the angle
[a] 7. 5. POR = EBC, and POQ = DAC [a]; then draw three right-lines to touch the circle in the points
[b] 21. 5. P, Q and R [b], *and the thing is done.*

For, if PQ be drawn, the angles SQP and SPQ,
[c] Ax. 2. 1. and 6. 3. will be less than the two right-angles SQO and SPO [c]; and so PS and QS, not being parallels [d],
[d] Cor. 2. to 7. 1. and Cor. to 5. 1. they will meet each other [d]; therefore, as the like may be inferred with regard to PT and RT, *&c.* it is manifest that the three tangents form a triangle STH. Now, POR + T being = two right-
[e] Cor. 2. to 11. 1. angles [e] = ABC + EBC [f], and POR = EBC [g]; thence will T = ABC: and, by the same argu-
[f] 1. 1. [g] Constr. ment, S = BAC; whence also H = C.

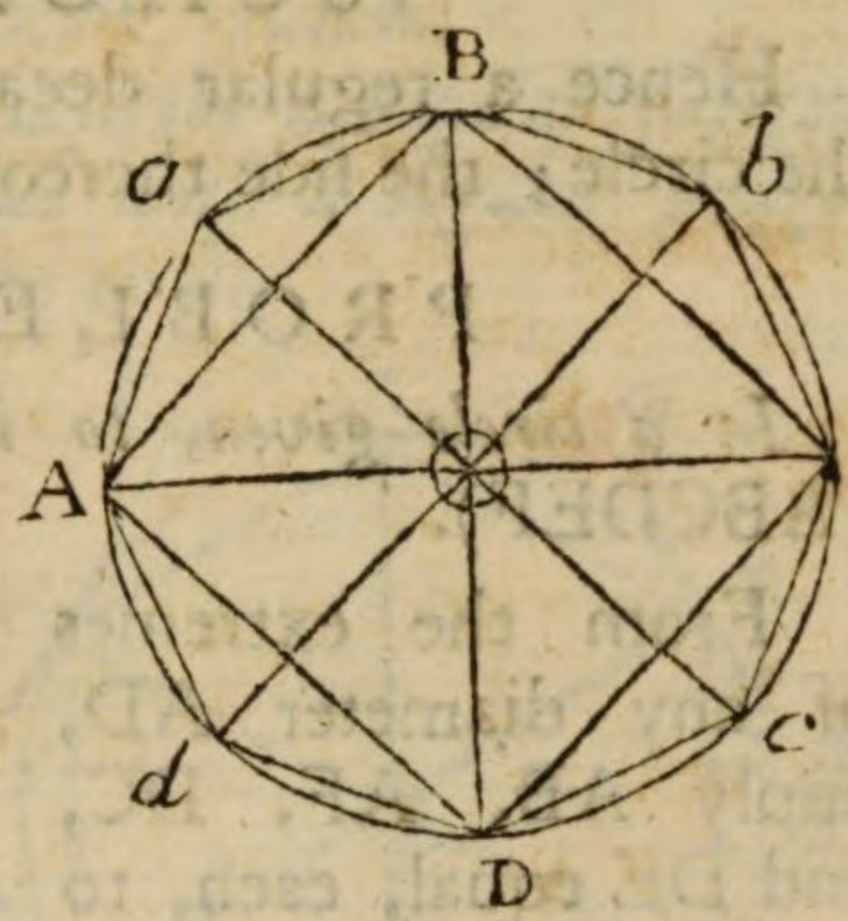
PROBLEM XXVII.

In a circle given (ABCD) *to inscribe a square.*

[h] 3. 5. Draw two diameters AC and BD perpendicular to each other [h]; then draw AB, BC, CD and DA; so shall ABCD be a square inscribed in the circle.

For,

For, the angles AOB, BOC, DOC and DOA (as well as the ſides OA, OB, OC, OD, containing them) being equal [i], the oppoſite ſides AB, BC, CD, DA will likewiſe be equal [k]: and the angles ABC, BCD, CDA, DAB, are all of them right-angles [l], and therefore are equal.



[i] Ax. 7. 1. and Def. 33. 1.
[k] Ax. 10. of 1.
[l] 13. 3.

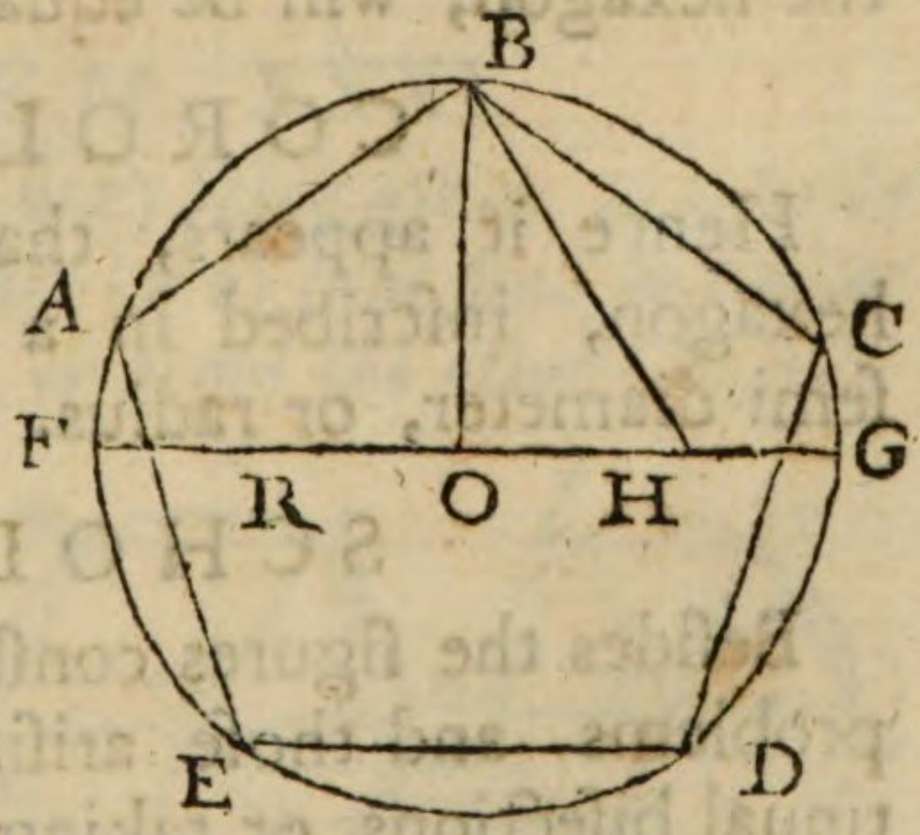
SCHOLIUM.

If two other diameters *ac*, *bd* be drawn (*by prob.* 5.) to biſect the angles AOB, BOC, a regular octagon A*a*B*b*C*c*D*d* may be inſcribed in the circle. And if all the angles at the center O, be again biſected, a regular polygon of ſixteen ſides, may in like manner be determined; and ſo on, at pleaſure.

PROBLEM XXVIII.

In a circle given (ABGE) *to inſcribe a regular pentagon.*

At the center O, upon the diameter FG, erect the perpendicular OB [m], meeting the circumference in B; divide OG in H (*by prob.* 19.) ſo that $OH^2 = GH \times OG$; that is, take $OR = \frac{1}{2}OF$, and RH = diſt. RB: Then draw BH; which will be equal to the ſide of the pentagon [n]; from whence the figure itſelf may be deſcribed.



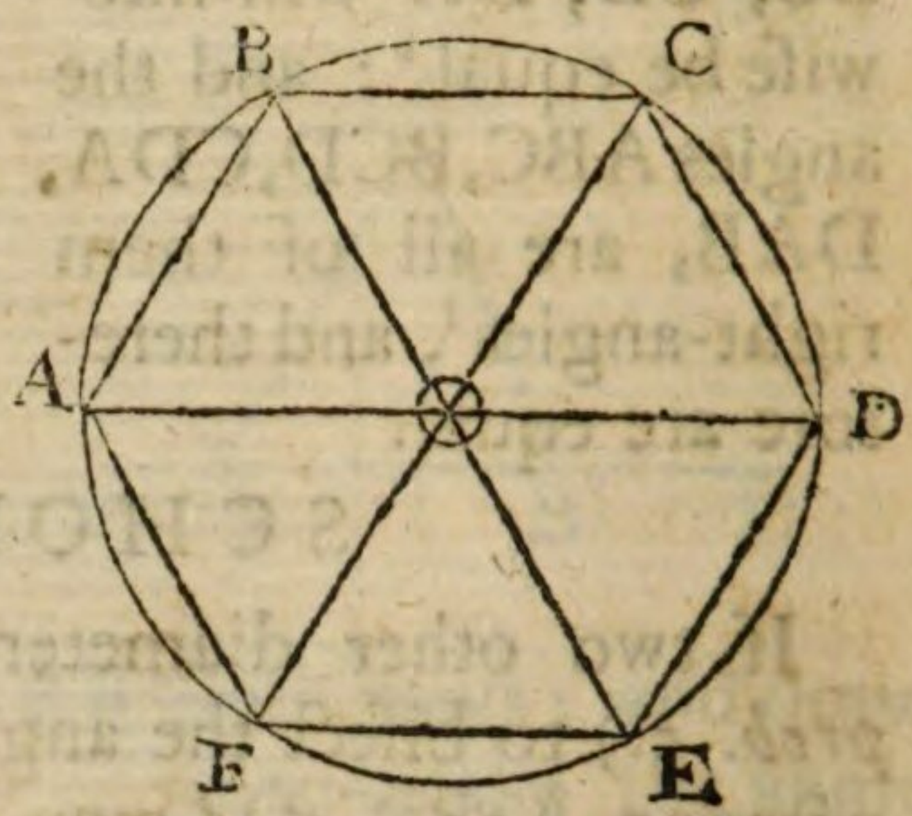
[m] 3. 5.
[n] 28. 3. and 8. 2.

SCHOLIUM.

Hence a regular decagon may be inſcribed in
[o] 28. 3. the circle; the ſide thereof being = OH [o].

PROBLEM XXIX.

In a circle given, to inſcribe a regular hexagon (ABCDEF).



From the extremes of any diameter AD, apply AB, AF, DC, and DE equal, each, to
[p] 20. 5. the radius AO [p]; then join B, C, and E, F; *and the thing is done.*

For, if the radii OB, OC, OE, OF be drawn; the triangles AOB and
[q] Conſtr. DOC, being equilateral [q], will alſo have the an-
[r] 14. 1. gle OAB = DOC [r]; whence AB is parallel (as
[s] Cor. to 8. 1. well as equal) to OC [s]; and conſequently BC and
[t] 26. 1. AO are likewiſe equal, and parallel [t]: Therefore, ſeeing the triangles AOB, BOC, COD, *&c.* are equilateral, and alike in all reſpects; not only the ſides, but alſo the angles ABC, BCD, *&c.* of
[u] Ax. 4 1. the hexagon, will be equal among themſelves [u].

COROLLARY.

Hence it appears, that the ſide of a regular hexagon, inſcribed in a circle, is equal to the ſemi-diameter, or radius.

SCHOLIUM.

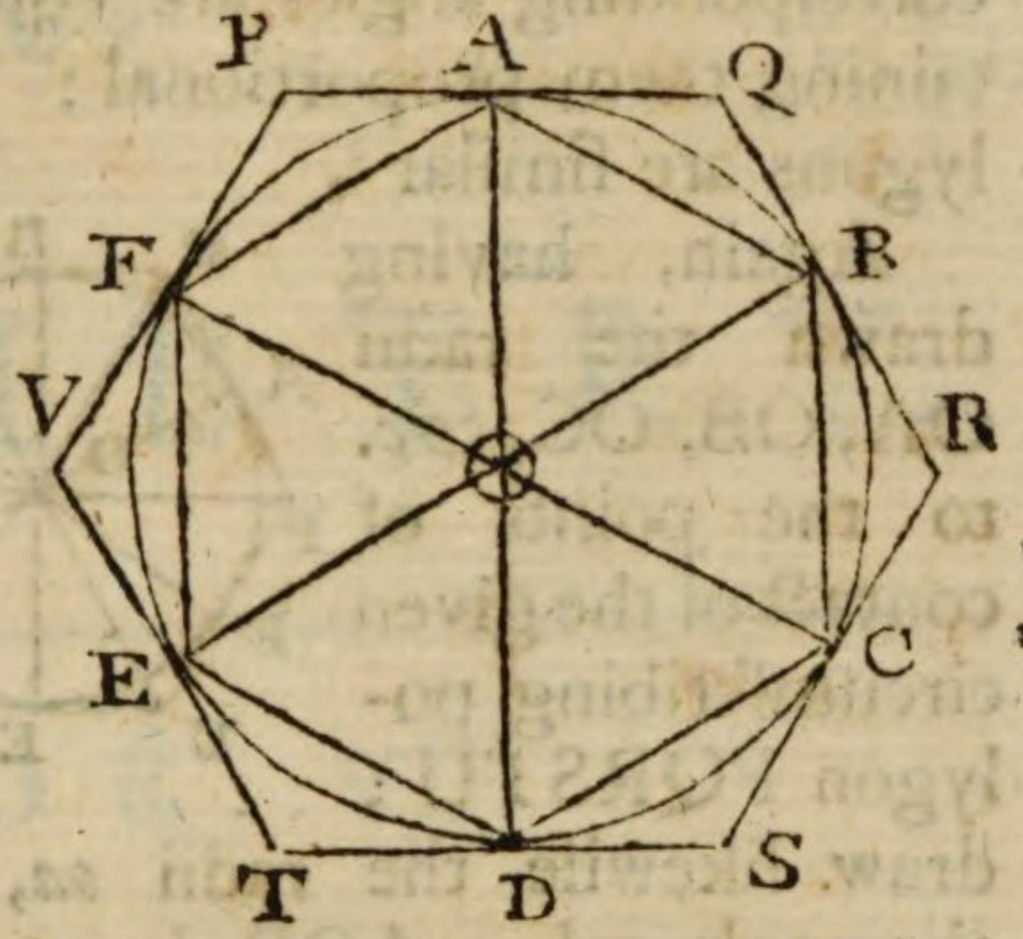
Beſides the figures conſtructed in the preceding problems, and thoſe ariſing from thence by continual biſections, or taking the differences, no other regular polygon can be deſcribed, from any known method, *purely geometrical*, by means of right-lines and circles only.

PRO-

PROBLEM XXX.

About a given circle to deſcribe a regular polygon, of the ſame number of ſides with a regular polygon (ABCDEF) *inſcribed in the circle.*

From the center O, to the angles of the inſcribed polygon, draw OA, OB, OC, &c. and perpendicular thereto draw PAQ, QBR, RCS [t], &c. interſecting [u] in P, Q, R, S, T, V; ſo ſhall PQRSTV be the polygon that was to be deſcribed.



[t] 3. 5.
[u] Proof of 26. 5.

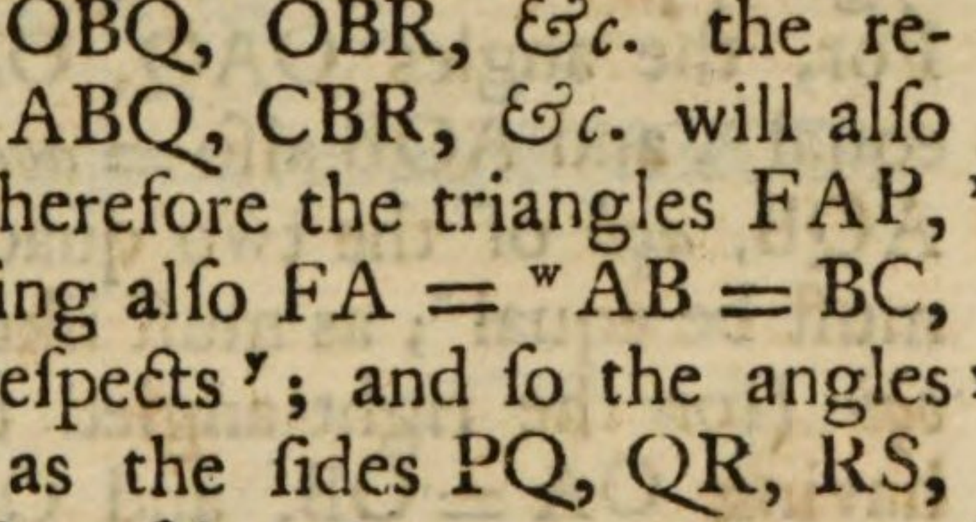
For, by taking away the equal [w] angles OAF, OAB, OBA, OBC, &c. from the equal (right) angles OAP, OAQ, OBQ, OBR, &c. the remainders FAP, BAQ, ABQ, CBR, &c. will alſo appear to be equal [x]: therefore the triangles FAP, ABQ, BRC, &c. (having alſo FA = [w] AB = BC, &c.) are equal in all reſpects [y]; and ſo the angles P, Q, R, &c. as well as the ſides PQ, QR, RS, &c. are equal among themſelves [z].

[w] Hyp.
[x] Ax. 5. 1.
[y] 15. 1.
[z] Ax. 4. 1.

PROBLEM XXXI.

Any two circles (ACE, ace) *being given, to deſcribe a polygon in, or about the one* (ace) *that ſhall be ſimilar to any polygon deſcribed in, or about the other* (ACE).

Firſt, having drawn the radii OA, OB, &c. to the angles of the given inſc. polyg. ABCDEF; make, at the center *o*, the angle *aob* = AOB, *boc* = BOC [a], &c. Then, the chords *ab*, *bc*, *cd*, &c. being drawn, I ſay, the polygon *abcdef* will be ſimi-

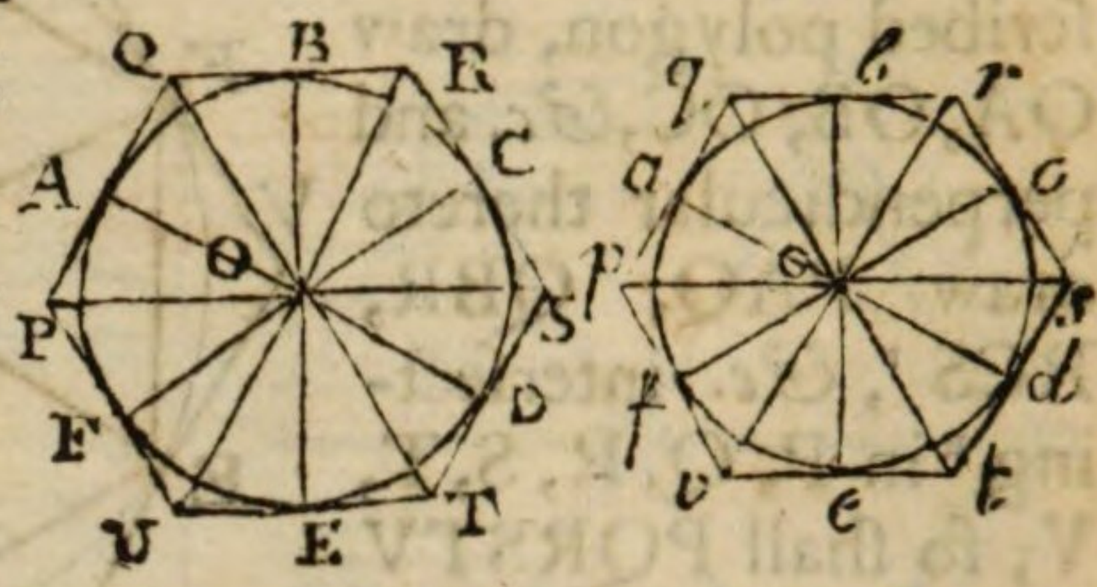


[a] 7. 5.

lar to the given one ABCDEF. For, the triangles AOB, *aob*; BOC, *boc*; &c. being equiangular [b], the angles ABC, *abc* must also be equal [c]; and [d] AB : *ab* (: : OB : *ob*) : : BC : *bc*. In the same manner, the other corresponding angles are equal, and the sides containing them proportional: Therefore the two polygons are similar [e].

[b] Cor. 1. to 10. 1. and 12. 1.
[c] Ax. 4. 1.
[d] 14. 4.
[e] Def. 14.

4. Again, having drawn the radii OA, OB, OC, *&c.* to the points of contact of the given circumscribing polygon PQRSTU; draw likewise the radii *oa*, *ob*, *oc*, &c. making the angle *aob* = AOB, *boc* = BOC [f], &c. perpendicular to which [g] draw *pq*, *qr*, *rs*, &c. so shall the polygon *pqrstv* be similar to the given one PQRSTU. For, the angles OAQ, OBQ, *oaq*, *obq*, being all equal [h], and AOB also = *aob* [i]; the remaining angles AQB, *aqb* of the two quadrilaterals AOBQ, *aobq* must be equal [k]; as must likewise their halves OQB, *oqb* (for the right-angled triangles OAQ, OBQ, having OA = OB, and OQ common, have also OQA = OQB [l]). In the same manner is ORB = *orb*, &c. whence, the triangles POQ, *poq*; QOR, *qor*, &c. being equiangular, it follows [m] that PQ : *pq* (: : OQ : *oq*) : : QR : *qr*: And so of the rest. Therefore PQRSTU and *pqrstv* are similar [n].



[f] 7. 5.
[g] 3. 5.
[h] Ax. 7. 1.
[i] Constr.
[k] 11. 1. and Ax. 5.
[l] 16. 1.
[m] 14. 4.
[n] Def. 14. 4.

COROLLARY.

It appears from hence, that the similar inscribed polygons, as well as the circumscribing ones, are in proportion, as the squares of the radii of their respective circles. For, in the former case, $AO^2 : ao^2$: : [o] $AB^2 : ab^2$: : [p] ABCDEF : *abcdef*; and, in the latter, [o] $AO^2 : ao^2$: : $PO^2 : po^2$: : $PQ^2 : pq^2$: : [p] PQRSTU : *pqrstv*.

[o] 14. 4. and Cor. to 11. 4.
[p] 26. 4.

The End of the FIFTH BOOK.

ELEMENTS

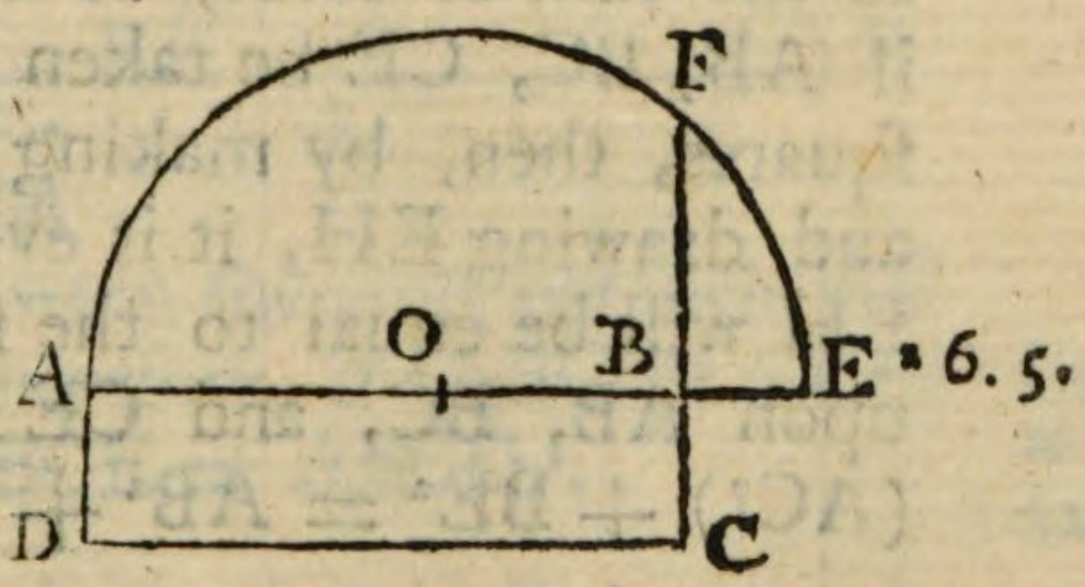
OF

GEOMETRY.

BOOK VI.

PROBLEM I.

To make a ſquare equal to a given rectangle (ABCD).



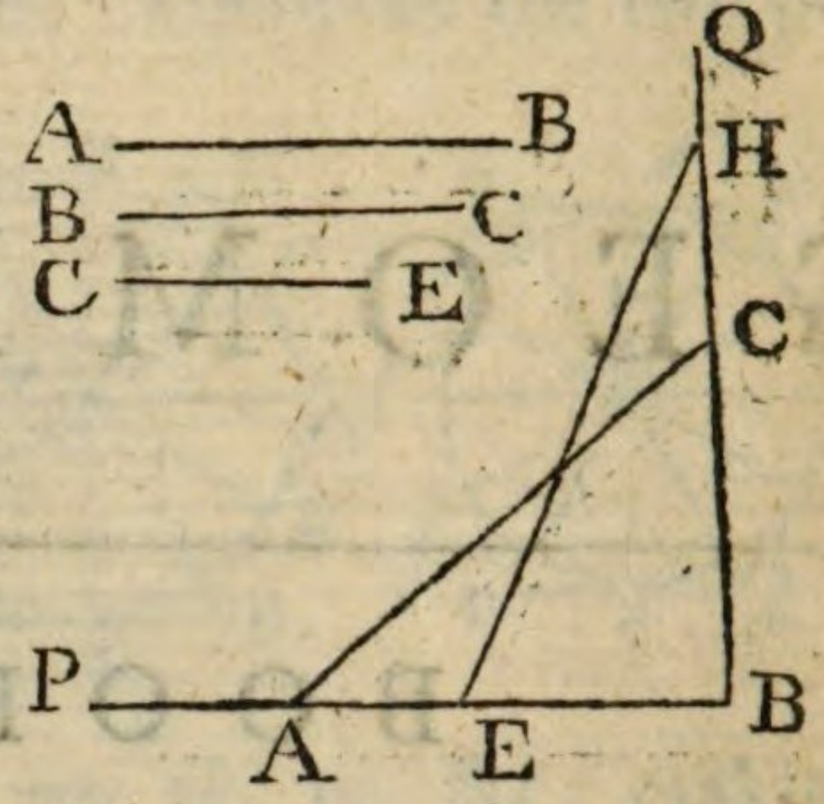
IN one ſide AB of the rectangle, produced, take BE = the other ſide BC; biſect AE in O [a]; and from the center O, at the diſtance of OA, or OE, let a ſemi-circle AFE be deſcribed; and let CB be produced to meet the circumference thereof in F: then a ſquare deſcribed on BF (*by* 10. 5.) will be equal to the given rectangle ABCD [b].

[a] 6. 5.

[b] Cor. to 19. 4.

PROBLEM II.

To make a ſquare equal to the ſum of two given ſquares.



Let AB and BC be the ſides of the two given ſquares.

Draw two indefinite lines BP, BQ, at right-angles to each other [c]; in which take BA = BA, BC = BC, and join A, C; then a ſquare deſcribed on AC *(by* 10. 5.*)* will be equal to the ſum of the two ſquares deſcribed upon AB and BC [d].

[c] 3. 5.

[d] 8. 2.

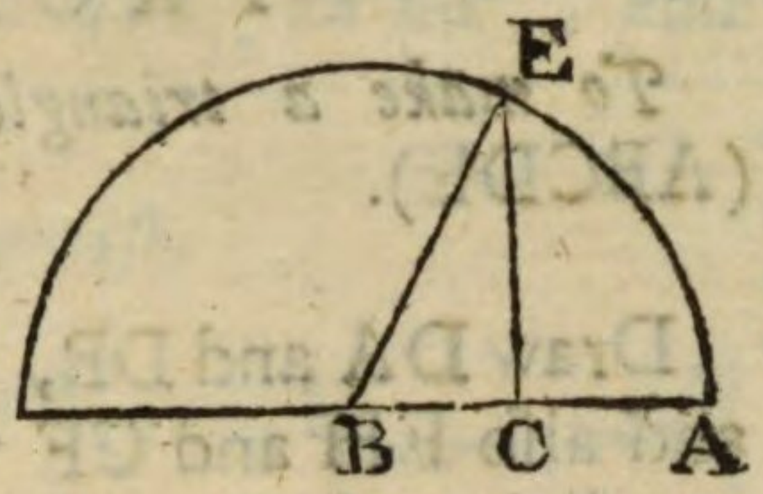
SCHOLIUM.

In the ſame manner a ſquare may be made equal to the ſum of three, or more, given ſquares: for if AB, BC, CE be taken as the ſides of the given ſquares, then, by making BH = AC, BE = CE, and drawing EH, it is evident that a ſquare upon EH will be equal to the ſum of the three ſquares upon AB, BC, and CE; or that, $EH^2 = BH^2 (AC^2) + BE^2 = AB^2 + BC^2 + CE^2$.

PROBLEM III.

To make a ſquare equal to the difference of two given ſquares.

Let



Let AB and BC (taken in the ſame ſtrait line) be equal to the ſides of the two given ſquares.

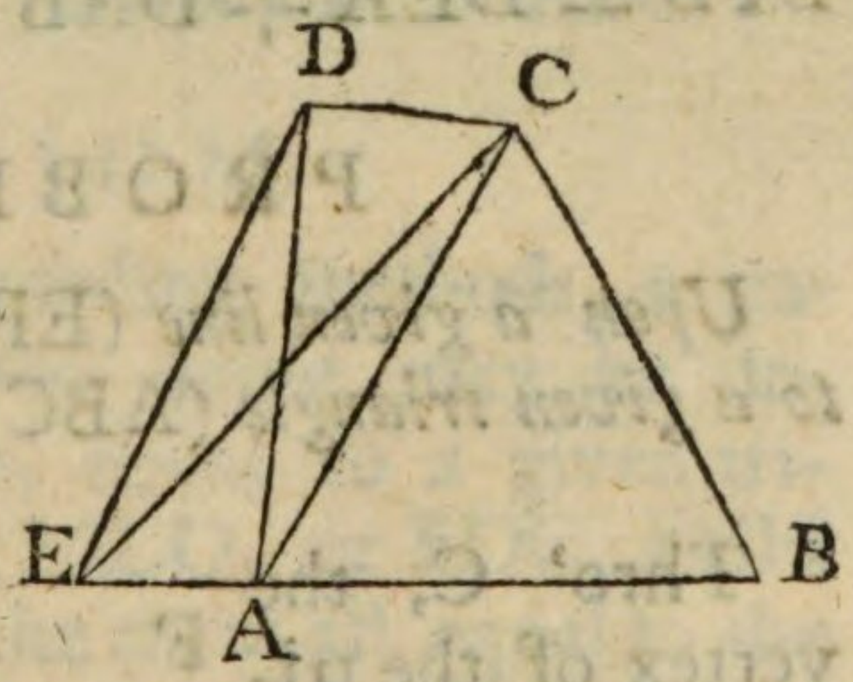
Upon the center B, with the radius BA, let a circle be deſcribed, and make CE perpendicular to BC [e], meeting the circumference thereof in E: ſo ſhall a ſquare deſcribed on CE (*by* 10. 5.) be equal to BE^2 (BA^2) — [f] BC^2.

[e] 3. 5.

[f] Cor. to 8. 2.

PROBLEM IV.

To make a triangle equal to a given quadrilateral (ABCD).



Draw the diagonal AC, alſo draw DE parallel to AC [g], meeting BA produced in E, and join CE; then will the triangle BCE = the given quadrilateral ABCD.

[g] 9. 5.

For, the triangles ACE, ACD, being upon the ſame baſe AC, and between the ſame parallels AC and ED, are equal [h]; therefore, if ABC be added to each, then alſo will BCE = ABCD [i].

[h] Cor. 1. to 2. 2.

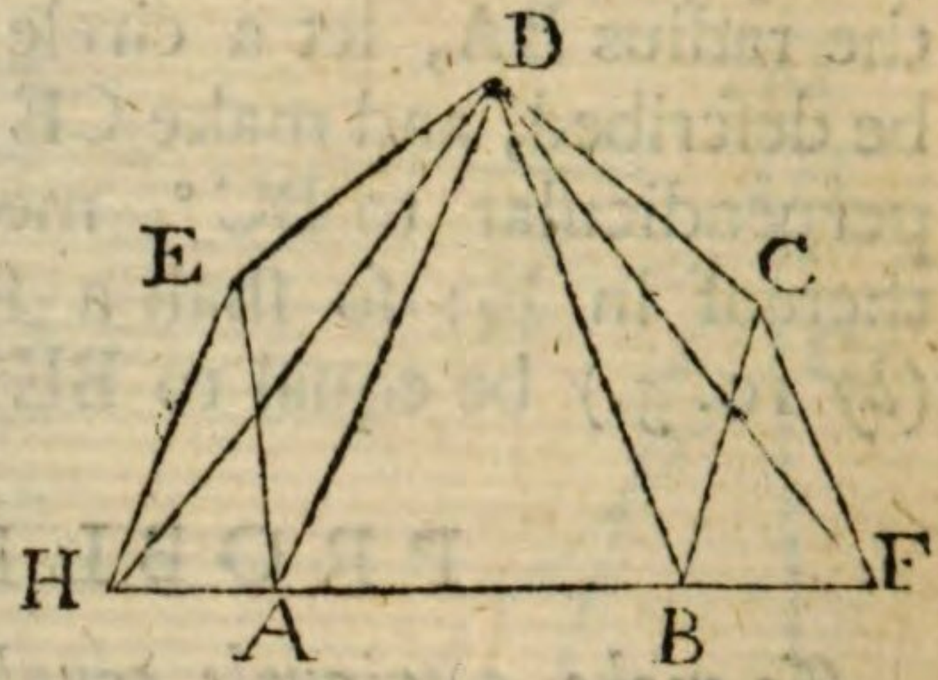
[i] Ax. 4. 1.

PROBLEM V.

To make a triangle equal to a given pentagon (ABCDE).

Draw DA and DB, and also EH and CF parallel to them [k], meeting AB produced in H and F; then draw DH and DF; so shall the triangle DHF = the pentagon ABCDE.

[k] 9. 5.



For the triangle DHA is = DEA [l], and DFB = DCB [l]; therefore DHF (= DHA + DAB + DFB = DEA + DAB + DCB) = ABCDE [m].

[l] Cor. 1. to 2. 2.

[m] Ax. 4. 1.

PROBLEM VI.

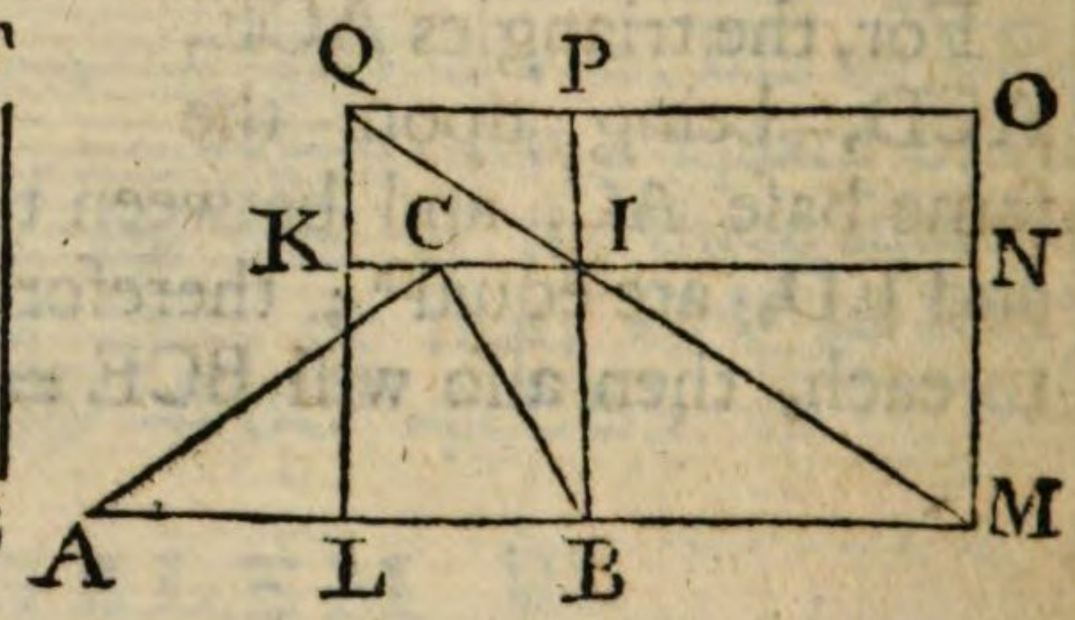
Upon a given line (EF), *to make a rectangle equal to a given triangle* (ABC).

Thro' C, the vertex of the triangle, draw KN parallel to the base AB [n]; and bisect AB with the perpendicular LQ [o], meeting KN in K; also draw BP perpendicular to AB [p], interfecting KN in I; then in AB, produced, take BM = EF, and draw MIQ cutting LQ in Q; draw QO and MO, parallel to AM and LQ [n], meeting each other in O: then will INOP be the rectangle required.

[n] 9. 5.

[o] 6. 5.

[p] 3. 5.

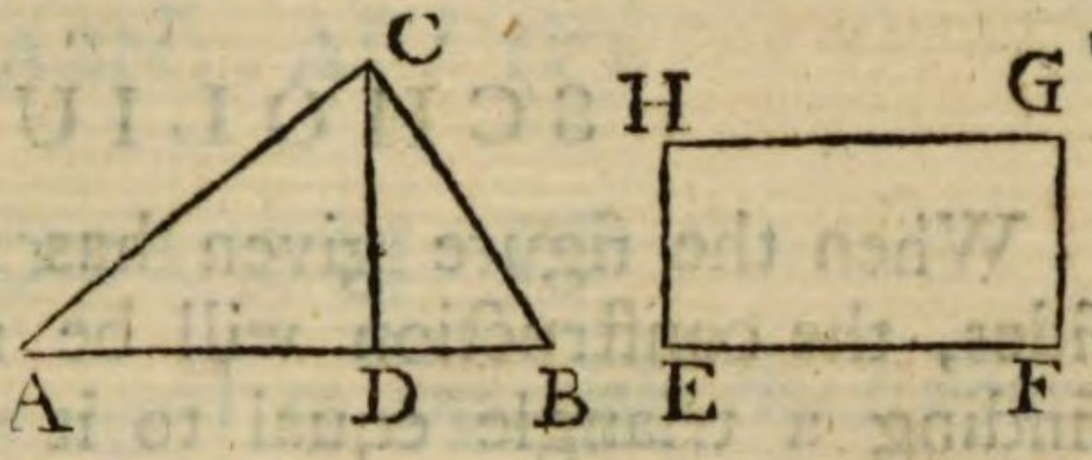


For,

For, it is evident, that LI, IO and LO are all rectangles [q]: therefore IN = BM [r] = EF [s], and IO = LI [t] = ABC [u].

[q] Cor. to 24. 1.
[r] 24. 1.
[s] Constr.
[t] 3. 2.
[u] Cor. 2. to 2. 2.

The same otherwise.



From the vertex C, upon the base AB, let fall the perpendicular CD [w]; make EH perpendicular to EF [x], and equal to a fourth-proportional to 2EF, AB, and CD [y]: then the rectangle EG contained under EF and EH will be equal to the triangle ABC.

[w] 4. 5.
[x] 3. 5.
[y] 13. 5.

For, since, by construction, 2EF : AB : : CD : EH, therefore is 2EF × EH = AB × CD [z], and consequently EF × EH = ½AB × CD = ABC [a].

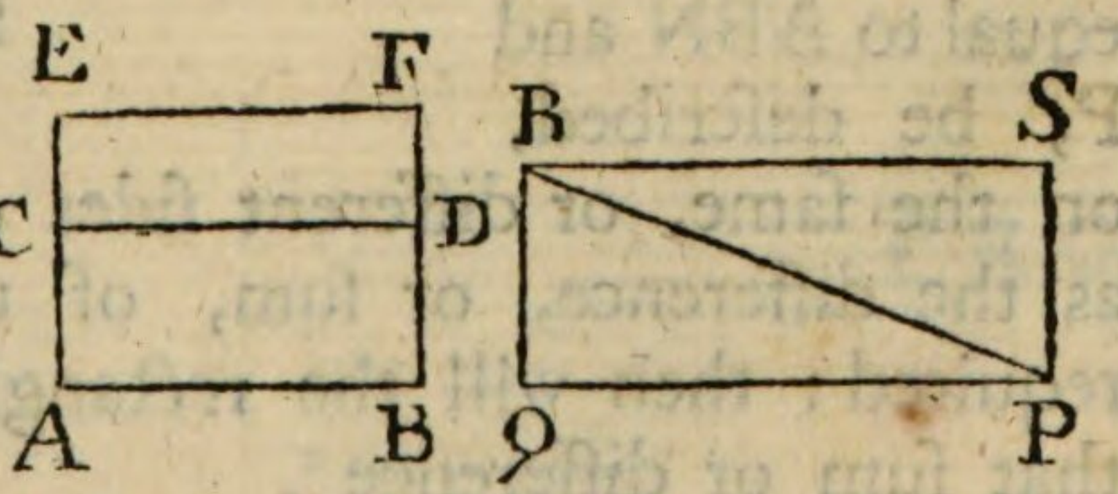
[z] 10. 4.
[a] Cor. 2. to 2. 2.

SCHOLIUM.

By either of the two preceding methods, a parallelogram having a given angle may be described, upon a given line, equal to a given triangle; if, instead of MBP, MLQ, or BDC, FEH being right-angles, you make them all equal to the angle given: the rest of the construction being the same.

PROBLEM VII.

Upon a given line (AB) *to describe a rectangle equal to a given right-lined figure* (PQRS).



Let the given figure be divided into triang. PQR, PRS: and upon the given line AB (by the precedent) let a rectangle ABDC, equal to the triangle PQR, be

be described; also upon CD make the rectangle CDFE equal to the triangle PRS: so shall ABEF, which is a rectangle (because both ACE, and BDE are continued right-lines [b]) be = PQR + PRS [c] = PQRS; *which was to be done.*

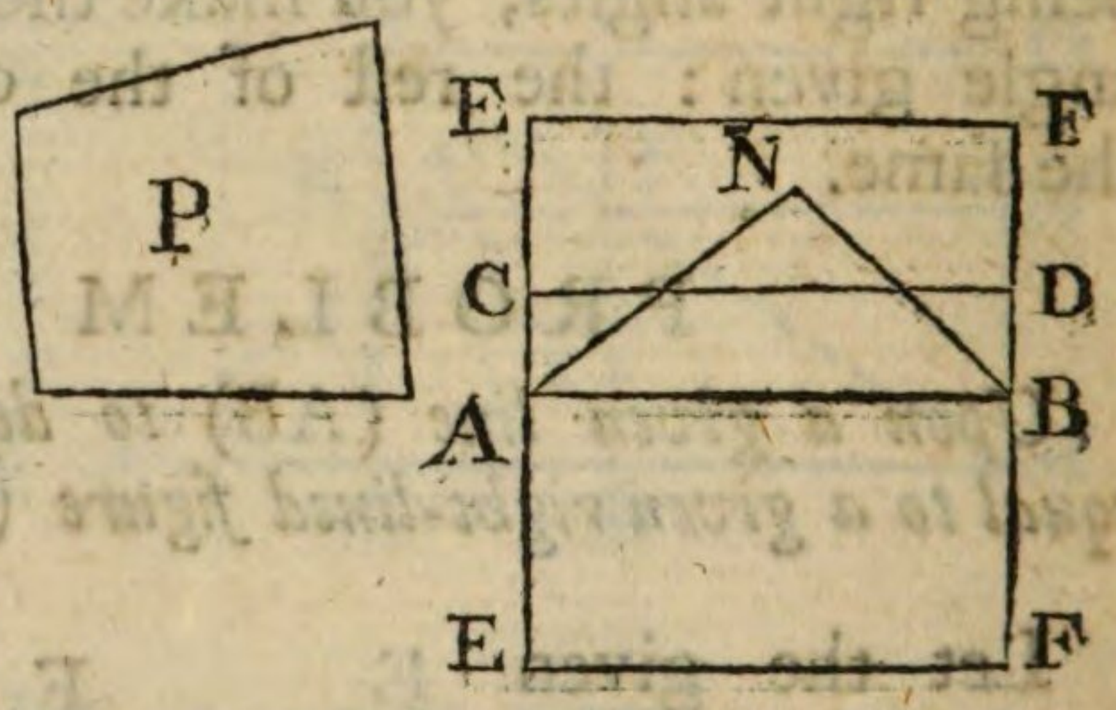
[b] 2. 1.
[c] Ax. 3. and 4. 1.

SCHOLIUM.

When the figure given has not more than five sides, the construction will be more easy, by first finding a triangle equal to it (*by prob.* 4. *or* 5.) and then making a rectangle equal to that triangle. But if the figure be a rectangle, the easiest way of all, will be to take a fourth-proportional BF to the given line AB and the two sides PQ and PS of the given rectangle (*by* 13. 5); which fourth-proportional will be the altitude of the rectangle required. For, since AB : PQ : : PS : BF (*by Constr.*) therefore (*by* 10. 4.) AB × BF = PQ × PS.

PROBLEM VIII.

To describe a rectangle equal to the sum, or difference of two given right-lined figures.



Let the two given figures be ABN and P.

By the precedent, let two rectangles AD and AF, respectively equal to ABN and P, be described, on the same, or different sides of AB, according as the difference, or sum, of the two figures is required: then will the rectangle CF be equal to that sum or difference [e].

[e] Ax. 4. or 5. 1.

COROL-

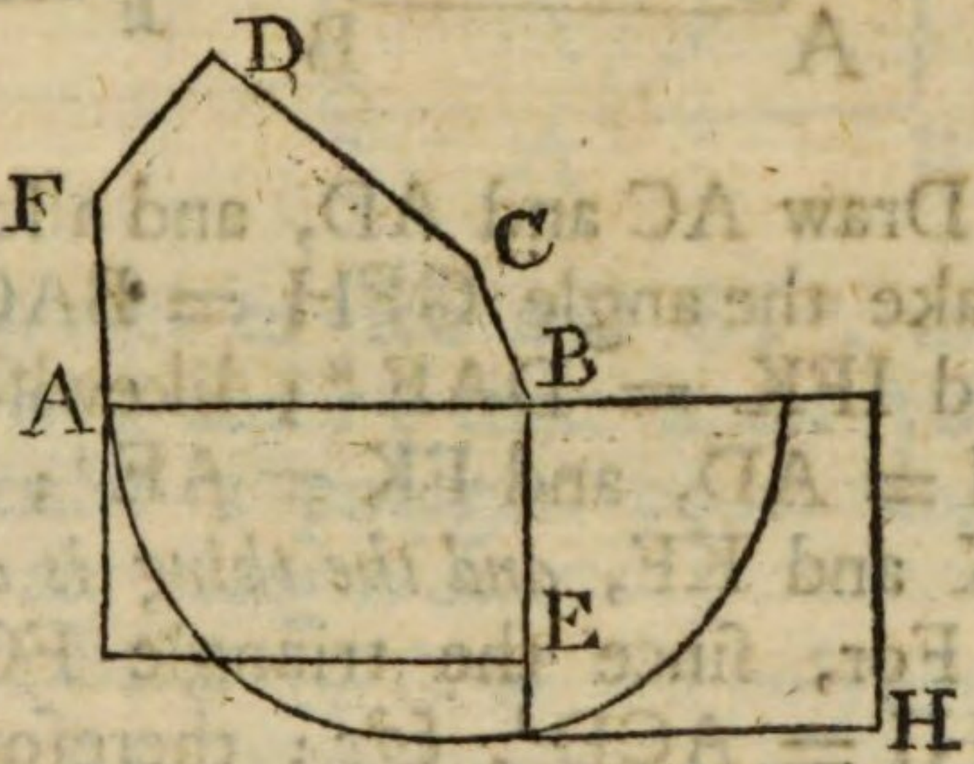
COROLLARY.

Hence two lines having the same ratio with two given right-lined figures, are determined: for AC : AE : : AD (ABN) : AF [f] (P). [f] 7. 4.

PROBLEM IX.

To make a square equal to any right-lined figure given (ABCDF).

Upon AB describe a rectangle AE equal to ABC DF [g]; then make a square BH equal to that rectangle [h], *and the thing is done.*



[g] 7. 6.

[h] 1. 6.

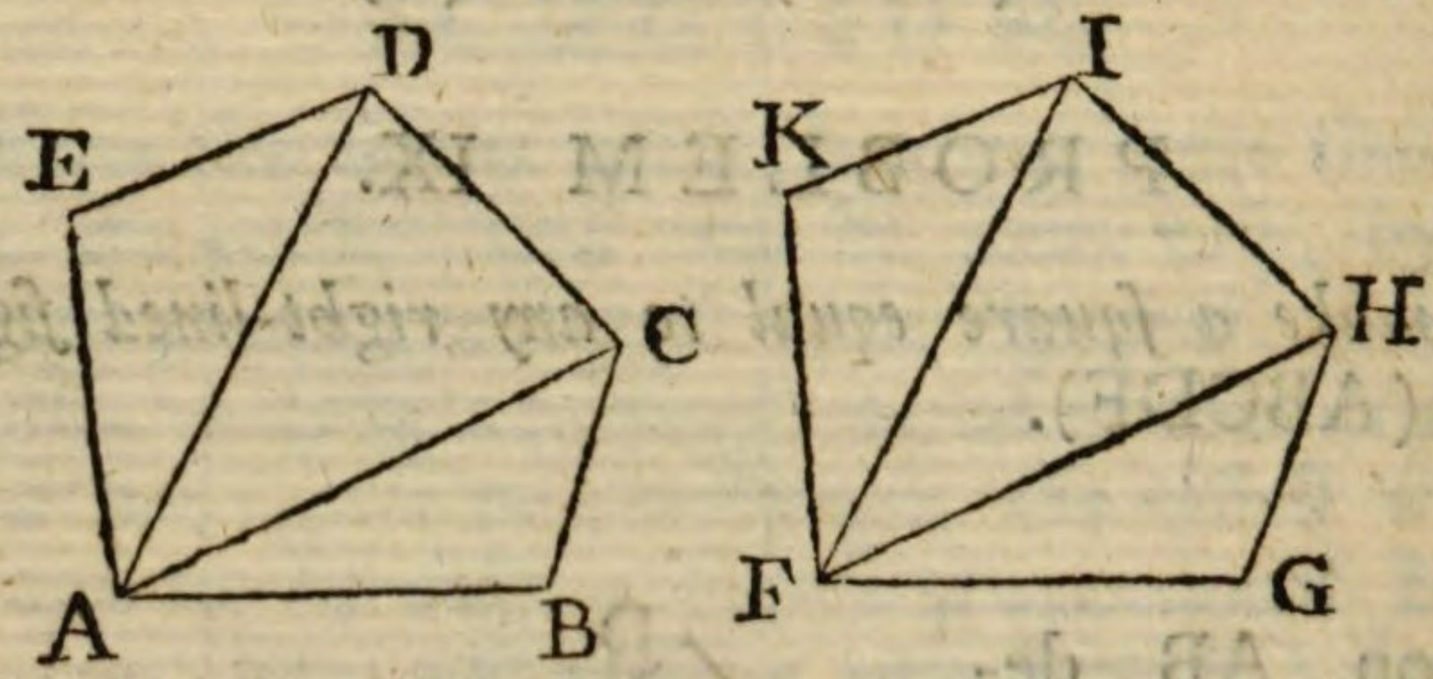
SCHOLIUM.

After the same manner (*from prob.* 8.) a square may be described equal to the sum, or difference, of any two given right-lined figures.

PRO-

PROBLEM X.

To describe a figure (FGHIK) *equal, and similar to a given right-lined figure* (ABCDE).



[i] 2. 5. Draw AC and AD, and also FG equal to AB [i]; [k] 7. 5. make the angle GFH = BAC [k], HFI = CAD, and IFK = DAE [k]; likewise make FH = AC, FI = AD, and FK = AE [i]; then draw GH, HI, IK and KF, *and the thing is done.*

[l] Ax. 10. 1. For, since the triangle FGH = ABC [l] and FHI = ACD [l], *&c*; therefore is the whole polygon FGHIK, also, equal to the whole polygon [m] Ax. 4. 1. ABCDE [m].

Moreover, these equal triangles being also equiangular [l], it is manifest, that G = B, GHI = BCD [m], HIK = CDE, and so on; therefore, FG being also = AB, GH = BC, HI = CD [l], *&c.* the two polygons ABCDE, FGHIK are similar to each [n] Def. 14. 4. other [n].

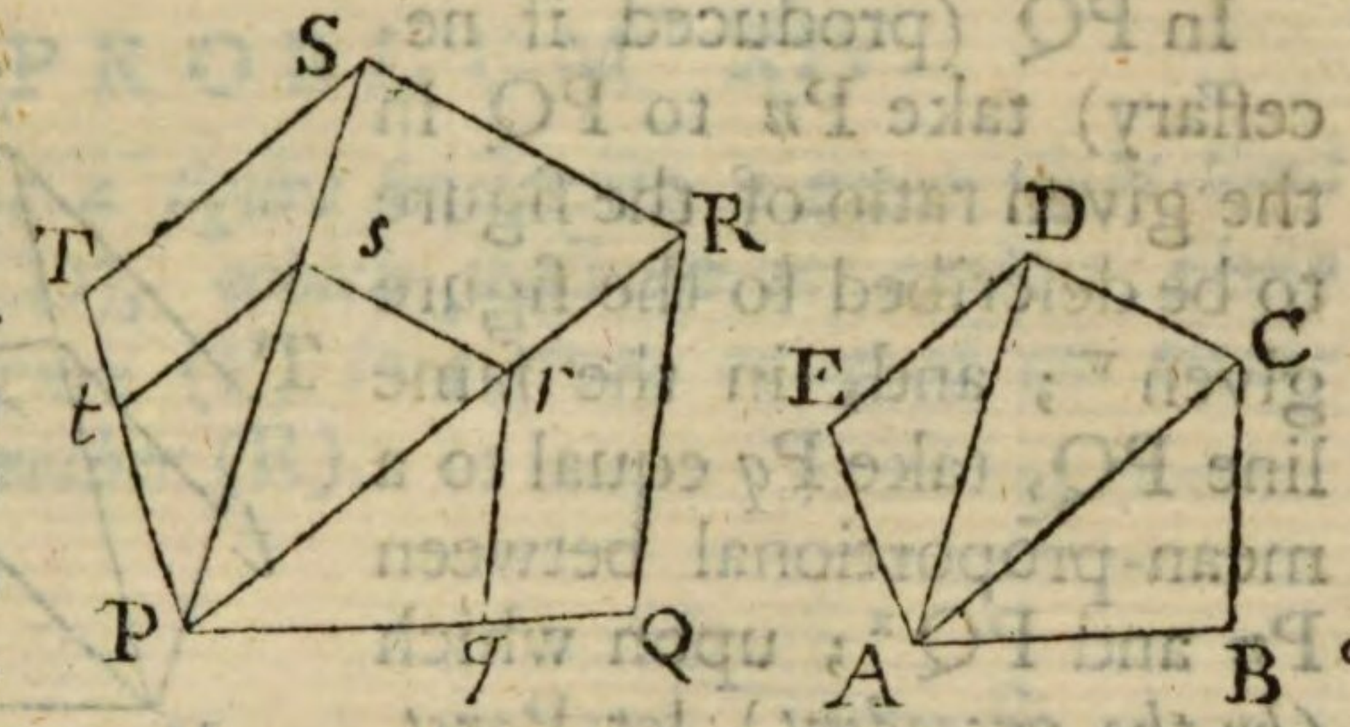
SCHOLIUM.

The figure FGHIK may be otherwise constructed, by making the triangles FGH, FHI, *&c.* respectively equilateral to ABC, ACD, *&c.* as is evident from 14. 1. and Ax. 4.

PRO-

PROBLEM XI.

Upon a given line (AB) *to describe a figure* (ABCDE) *similar to a given right-lined figure* (PQRST).



Draw PR and PS, and in PQ (produced if needful) take $Pq = AB$; draw *qr* parallel to QR °, meeting PR in *r*; also draw *rs* and *st*, parallel to RS and ST °, interfecting PS and PT in *s* and *t*; then upon AB, *by the precedent*, deſcribe a polygon equal and ſimilar to *Pqrst*; *and the thing is done.*

For, ſince any angle BCD (*qrs*) of the polygon ABCDE, is equal to its correſpondent QRS [p]; and alſo CB (*qr*) : CD (*rs*) : : RQ : RS [q], therefore the two polygons ABCDE, PQRST are like to each other [r].

[p] Cor. 1. to 7. 1.
[q] 21. 4.
[r] Def. 14. of 4.

SCHOLIUM.

This laſt problem may be otherwiſe conſtructed, by making the triangles ABC, ACD, ADE equiangular to the triangles PQR, PRS, PST, reſpectively.

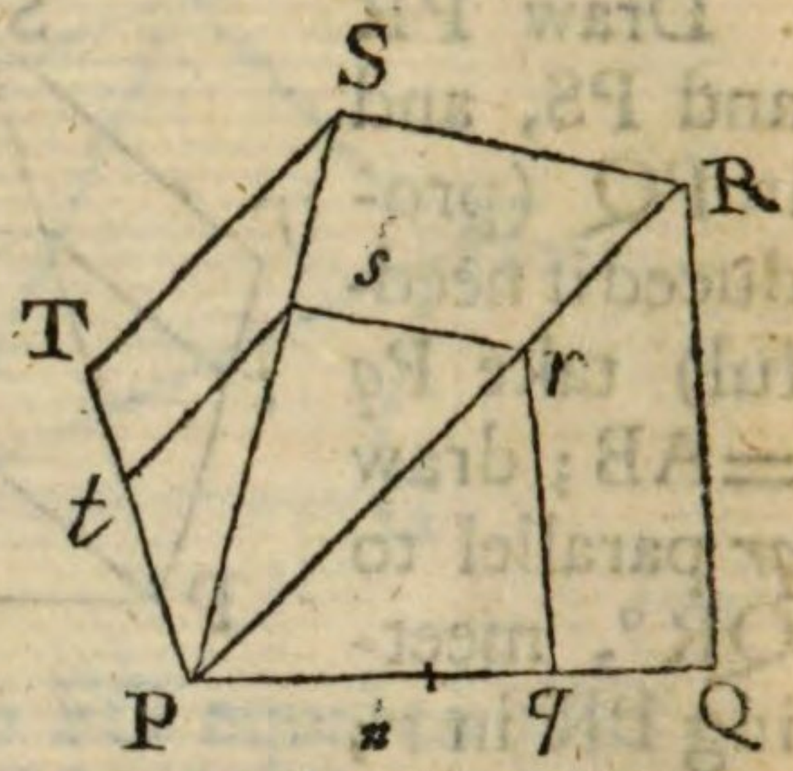
For, then the angle BCD being = QRS, CDE = RST [s], *&c*; and alſo BC : QR (: : AC : PR) : : CD : RS [t], *&c*. the two polygons muſt therefore be ſimilar to each other [u].

[s] Ax. 4. 1.
[t] 14. 4.
[u] Def. 14. of 4.

PRO-

PROBLEM XII.

To describe a figure, similar to a right-lined figure given (PQRST), *which shall be to it in a given ratio of one right-line to another.*



In PQ (produced if necessary) take P*n* to PQ in the given ratio of the figure to be described to the figure given [w]; and, in the same line PQ, take P*q* equal to a mean-proportional between P*n* and PQ [x]; upon which *(by the precedent)* let P*qrst*, similar to PQRST, be described, *and the thing is done.*

[w] 12. 5.

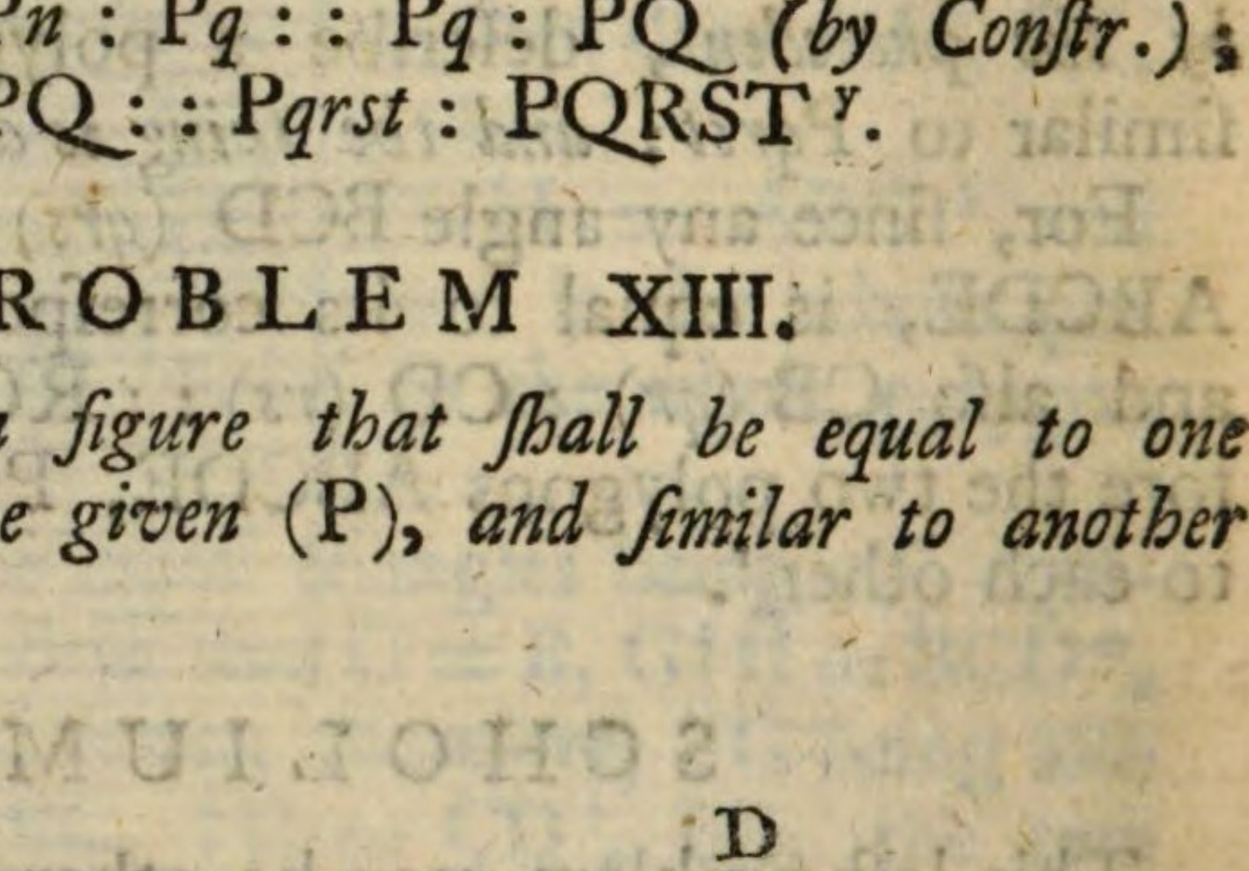
[x] 14. 5.

For, since P*n* : P*q* : : P*q* : PQ *(by Constr.)*; therefore P*n* : PQ : : P*qrst* : PQRST [y].

[y] 27. 4.

PROBLEM XIII.

To describe a figure that shall be equal to one right-lined figure given (P), *and similar to another* (ABCD).



Upon AB make the rectangle ABFG = ABCD [z], and upon AG make the rectangle AGNE = P [z]; in AB take AI equal to a mean-proportional between AB and AE [a]; and upon AI let AIKH be described similar,

[z] 7. 6.

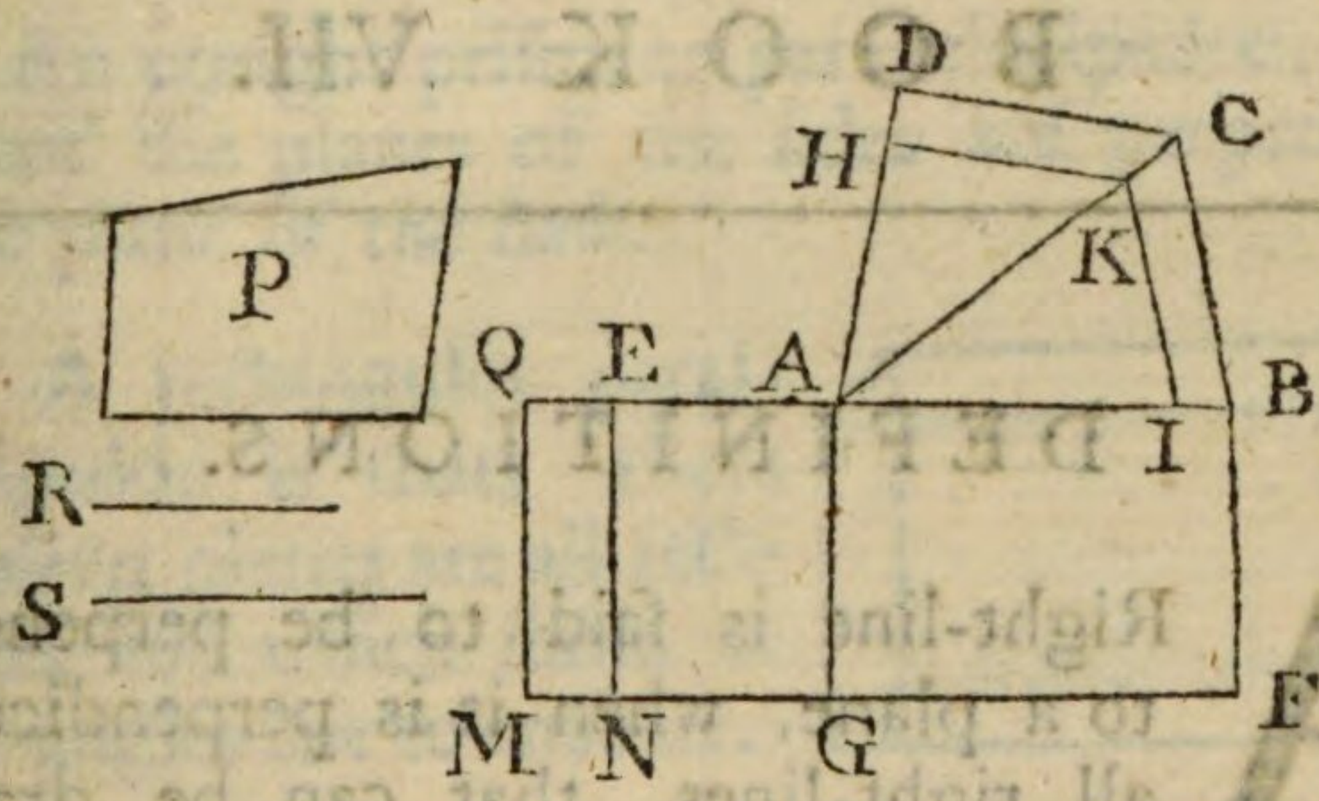
[a] 14. 5.

lar, and alike ſituate, to ABCD [b], *and the thing is done.* [b] 11. 6.

For ABCD (AF) : P (AN) : : AB : AE [c] (: : AB² : AB × AE [c] = [d] AI²) : : ABCD : AIKH [e]; and therefore AIKH = P [f]. [c] 7. 4. [d] Conſtr. and 10. 4. [e] 26. 4. [f] Ax. 4. 4.

PROBLEM XIV.

To deſcribe a figure ſimilar to a given right-lined figure (ABCD), *which ſhall be to another given right-lined figure* (P) *in a given ratio of one right-line* (S) *to another* (R).



Make the rectangle ABFG = ABCD [g], and the rectangle AGNE = P [g]; alſo in AE, produced, take AQ = a fourth-proportional to R, S and AE [h]; then, *by the precedent*, make AIKH ſimilar to ABCD, and equal to the rectangle AG × AQ: then will AIKH (AG × AQ) : P (AG × AE [i]) : : AQ : AE [k] : : S : R [i]; *which was to be done.* [g] 7. 6. [h] 13. 5. [i] Conſtr. [k] 7. 4.

The End of the SIXTH BOOK.

ELEMENTS
OF
GEOMETRY.

BOOK VII.

DEFINITIONS.

I. A Right-line is ſaid to be perpendicular to a plane, when it is perpendicular to all right-lines, that can be drawn in that plane, from the point on which it inſiſts.

2. One plane is ſaid to be perpendicular to another, when all right-lines drawn in the one, perpendicular to the common ſection, are perpendicular to the other.

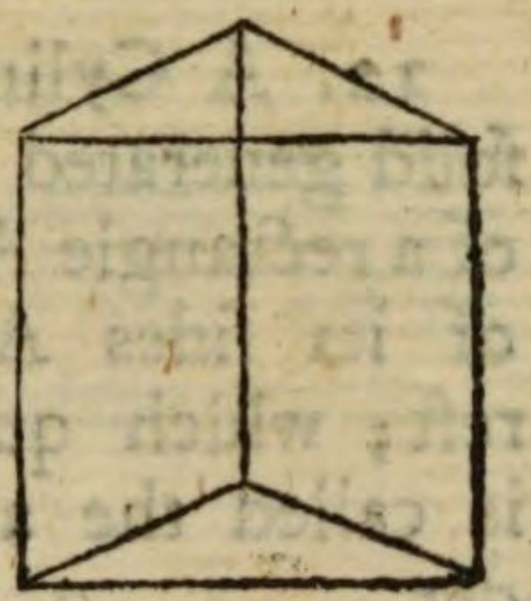
3. Paralliel planes are thoſe, which are every where equally diſtant, the one from the other.

4. A Solid is that, which has length, breadth, and thickneſs.

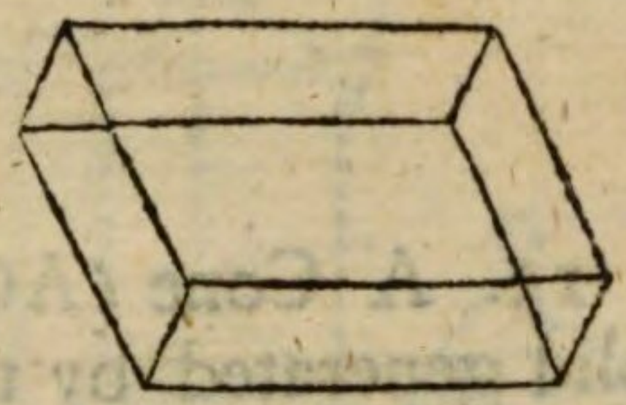
5. Similar ſolids are ſuch, as are bounded by an equal number of ſimilar planes.

6. A Priſm

6. A Priſm is a ſolid, whereof the planes of the ſides are parallelograms, and whereof the two ends, or oppoſite baſes, are plane, rectilinear figures, parallel to each other.

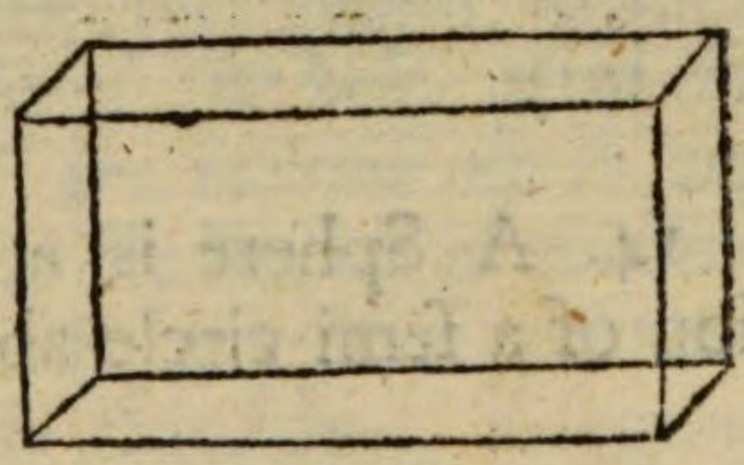


7. A Parallelepipedon is a ſolid bounded by ſix parallelograms, whereof the oppoſite ones are parallel, equal, and like to each other.

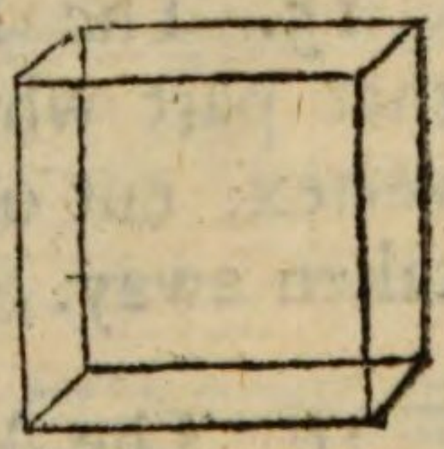


8. An upright priſm, or parallelepipedon, is that, whereof the planes of the ſides are perpendicular to the plane of the baſe.

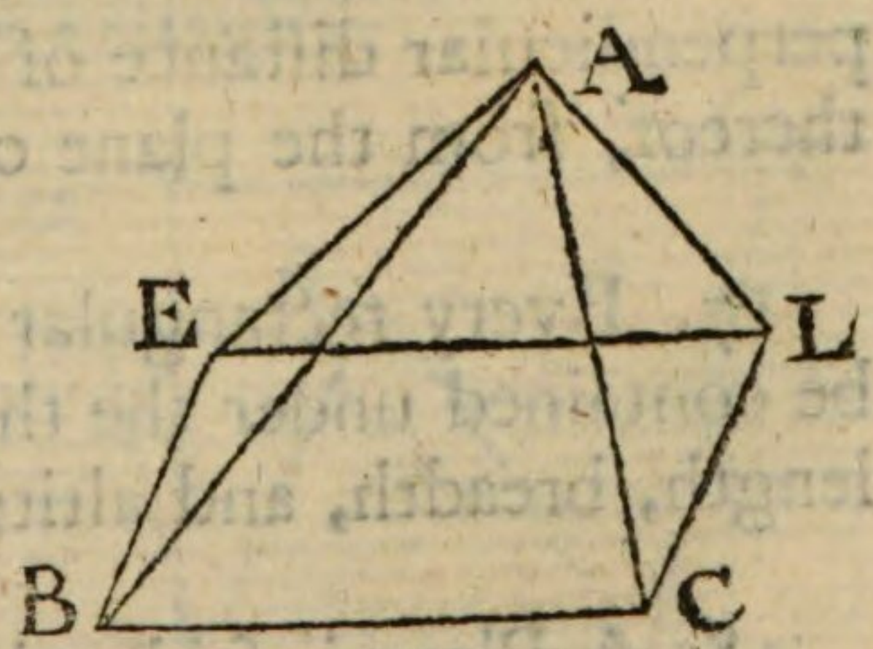
9. A rectangular parallelepipedon is that, whoſe bounding planes are all rectangles, and which ſtand at right-angles one to another.



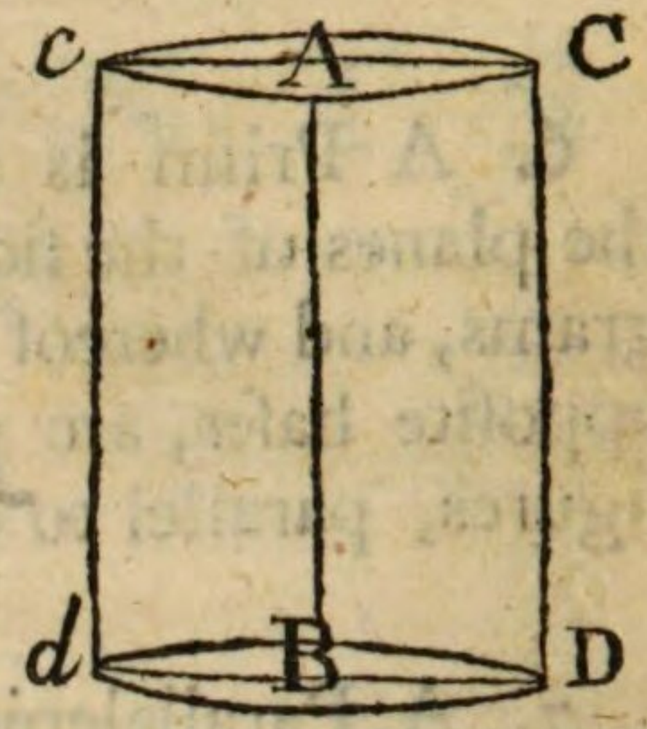
10. When all the bounding planes are ſquares, the parallelepipedon is called a cube.



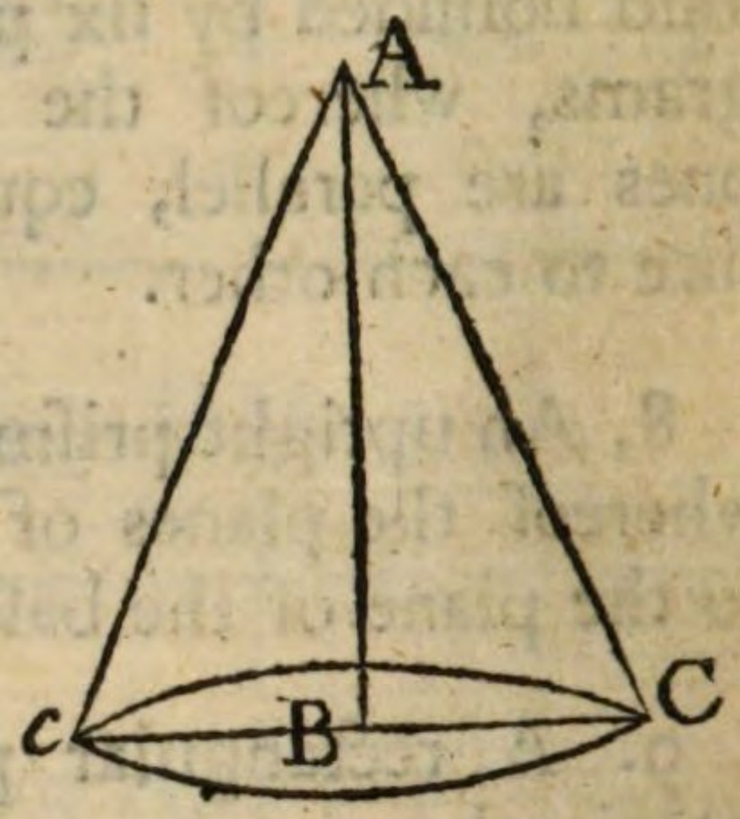
11. A Pyramid is a ſolid, whoſe baſe is any right-lined plane figure, and whoſe ſides are triangles, having all their vertices united in a point, above the baſe, called the vertex of the pyramid. Thus ABCLE repreſents a pyramid, whoſe vertex is A, and baſe BCLE.



12. A Cylinder (DC*cd*) is a ſolid generated by the rotation of a rectangle ACDB about one of its ſides AB, ſuppoſed at reſt; which quieſcent ſide AB is called the axis of the cylinder.



13. A Cone (AC*c*) is a ſolid generated by the rotation of a right-angled triangle ABC about its perpendicular AB, called the axis of the cone.



14. A Sphere is a ſolid generated by the rotation of a ſemi-circle about its diameter.

15. The Fruſtum of a pyramid, or cone, is that part which remains, when any part next the vertex, cut off by a plane parallel to the baſe, is taken away.

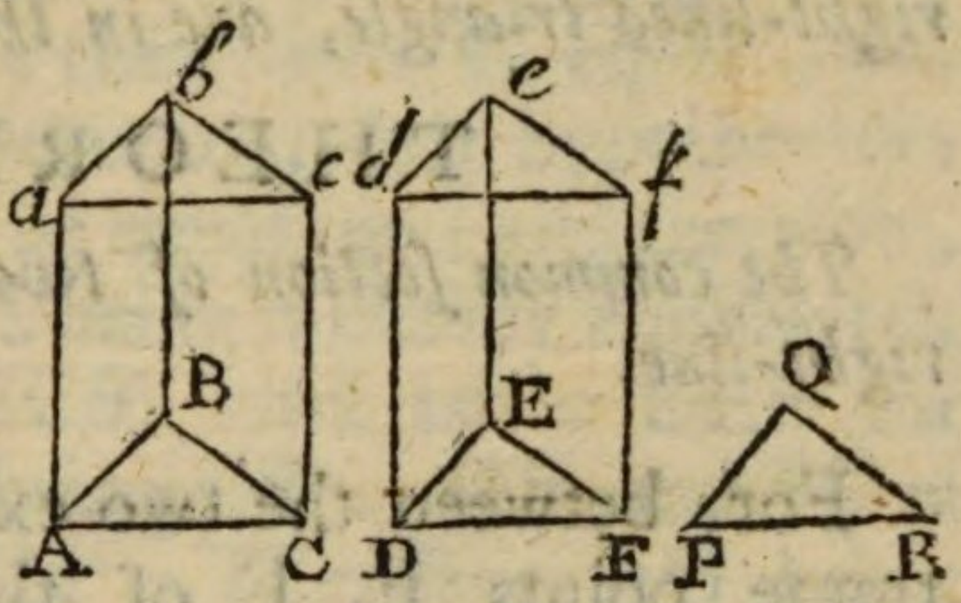
16. The Altitude of a pyramid, or priſm, is the perpendicular diſtance of the vertex, or upper plane thereof, from the plane of the baſe.

17. Every rectangular parallelepipedon is ſaid to be contained under the three right-lines that are the length, breadth, and altitude thereof.

18. A Plane is ſaid to be extended (or to paſs) by a right-line, when every part of the latter is placed in, or touched by, the former.

An AXIOM.

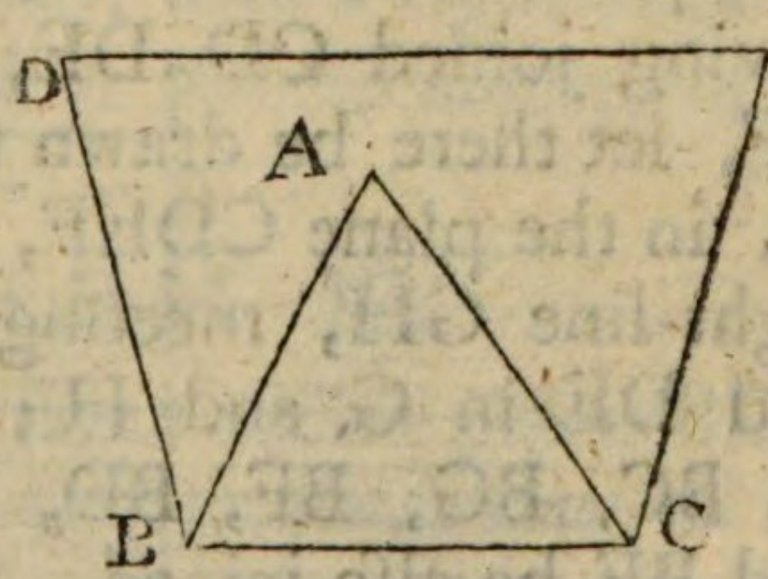
Upright priſms (A*abc*CBA, D*def*FED) of the ſame altitude, ſtanding upon baſes (ABC, DEF) equal and like to each other, are themſelves equal.



To ſee the evidence of this AXIOM *in the ſtrongeſt light, conceive a right-lined plane figure* PQR *to be formed, equal and like in all reſpects to the baſes* ABC, DEF *of the two priſms; upon which, conceive the priſms to be placed, one after another, ſo that their baſes may coincide therewith: Then, becauſe the planes of the ſides ſtand, in both caſes, perpendicular to the plane of the baſe, upon the ſame lines* PQ, QR, PR, *and are carried up to the ſame height, it is manifeſt, that the bounds of the two ſolids, when thus placed, have the very ſame poſition; and, conſequently, that the ſolids themſelves, occupying (ſucceſſively) the ſame identical ſpace, are equal the one to the other.*

A POSTULATE.

That by any two right-lines (AB, AC) meeting in a point, a plane may be extended.



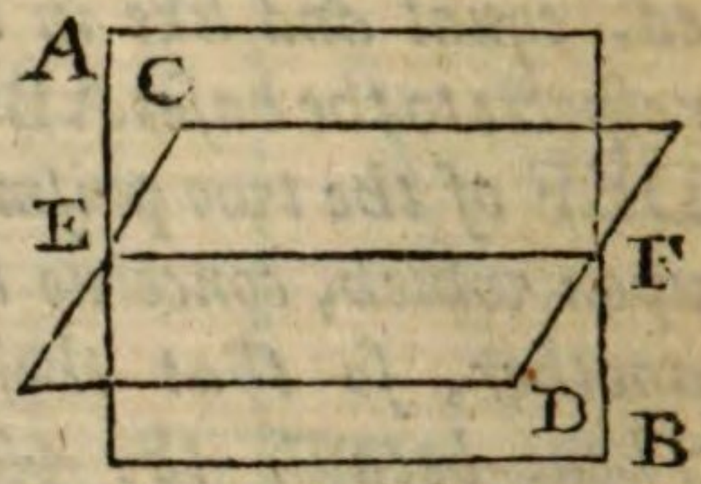
In order the better to comprehend the ſenſe and deſign of this Poſtulate, *let a plane* BDEC, *extended by the right-line joining the points* B *and* C, *be conceived to be revolved about upon that line, till it meets with, or takes in, the point* A; *then, the plane including, in that poſition, all*

the three points B, C, *and* A, *it alſo includes, or is extended by, the right-lines* AB, AC, BC, *joining thoſe points; which are in the ſame plane with their extremes (by Def.* 6. 1).

Hence it appears, that, by any three points, a plane may be extended; and that all the three ſides of any right-lined triangle, are in the ſame plane.

THEOREM I.

The common ſection of two planes (AB, CD) *is a right-line.*

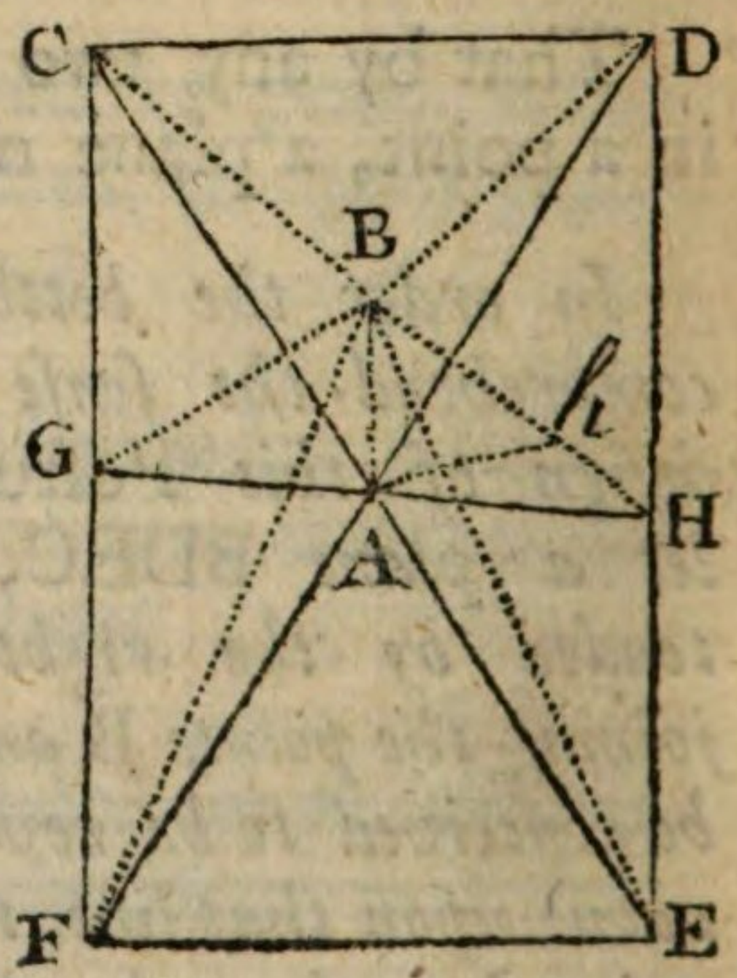


For, between the two extreme points E, F of the common ſection, let a right-line EF be drawn[a]; then, that line being in the plane AB[b], and alſo in the plane CD[b], it muſt, of conſequence, be the common ſection of them both.

[a] Poſt. 1.
[b] Def. 6. 1. and Ax. 8. 1.

THEOREM II.

If a right-line (AB) *be perpendicular to two other right-lines* (CE, DF) *cutting each other, at the common ſection* (A), *it will be perpendicular to the plane* (CDEF) *paſſing by thoſe two lines.*



Take AC, AD, AE, AF all equal to one another; and, having joined CD, DE, EF, CF, let there be drawn thro' A, in the plane CDEF, any right-line GH, meeting CF and DE in G and H; and let BC, BG, BF, BD, BH, and BE be alſo joined.

Becauſe

Becauſe AC[b] = AE = AD = AF, and CAF = DAE[c], therefore is CF = DE[d], and the angle FCA (or GCA) = DEA (or HEA); and ſo, GAC being likewiſe = HAE[c] and AC = AE[b], thence will AG = AH[e] and GC = HE.

[b] Conſtr.
[c] 3. 1.
[d] Ax. 10. 1.
[e] 15. 1.

Again, ſince the right-angled triangles CAB, DAB, EAB, FAB have their baſes all equal[f], and the perpendicular AB common, their hypothenuſes BC[d], BD, BE, BF will be equal too; and therefore, the triangles CBF, DBE being mutually equilateral, the angle FCB (or GCB) muſt be = DEB (or HEB[g]); whence, GC being alſo = HE, and BC = BE, thence is BG = BH[d]: Therefore, AG being likewiſe (as is proved above) = AH, and AB common, the angles GAB, HAB are equal, and conſequently right-angles[h]. In the ſame manner, AB is perpendicular to every other right-line drawn thro' A in the plane CDEF; *which was to be demonſtrated*[i].

[f] Hyp.
[g] 14. 1.
[h] Def. 8. of 1.
[i] Def. 1. of 7.

COROLLARY.

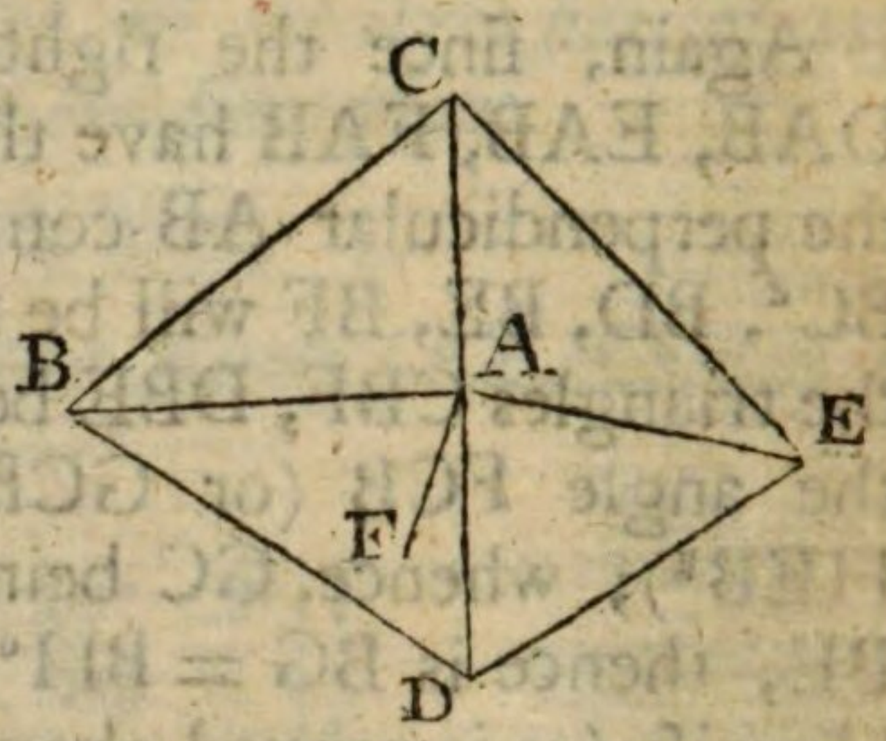
Hence it will appear, that, if one right-line (AB), meeting ſeveral others (AF, AE, *&c.*) in the ſame point (A), is perpendicular to them all; theſe laſt will be all in the ſame plane. Becauſe it is impoſſible for a right-line (A*b*) drawn from A, out of the plane (FEDC) of the two former of theſe, to be perpendicular to AB; ſeeing the angle BA*b* is leſs, or greater, than a right-angle (or BAH[k]), according as A*b* is poſited above, or below the ſaid plane FEDC[l].

[k] 2. 7.
[l] Ax. 2. 1.

THEOREM III.

If thro' any given point (A) *in a plane given* (BCD), *a line* (CD) *be drawn, and perpendicular to that line, at the ſame point* (A), *two other lines* (AB, AE) *be alſo drawn, the one* (AB) *in the plane given* (BCD), *and the other in any other plane* (CDE) *paſſing*

passing by the first line (CD); *then, I say, that a right-line* (AF) *drawn from the given point* (A), *at right-angles to the first perpendicular* (AB) *in the plane* (BAE) *of the two, will be perpendicular to the given plane* (BCD) *at the given point* (A).



For CA being perpendicular both to AB and AE[m], it will likewise be perpendicular to AF[n]; and so FA, being perpendicular to AB[m] (as well as to CA) is also perpendicular to the plane BCD, in which AB and CA are drawn[n].

[m] Hyp.
[n] 2. 7.

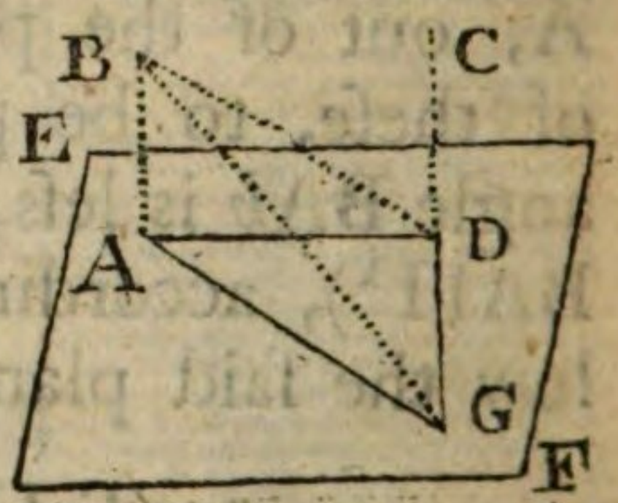
SCHOLIUM.

In this last Theorem, the manner of erecting a perpendicular to a plane, at a point given, is indicated, and the consistence of the *first definition*, of this book, evinced.

THEOREM IV.

Two right-lines (AB, CD), *perpendicular to the same plane* (EF), *are parallel to each other.*

Draw, in the plane EF, the right-line AD, and also DG perpendicular to AD; make DG = AB, and let AG, BG, and BD be drawn.



The triangles BAD, ADG, having AB = DG[o], AD common, and the angle BAD (= right-angle[p]) = ADG[o], will also have BD = AG[q]: And so, the triangles BDG, BAG being mutually equilateral, the angle[r] BDG must be = BAG = [p] right-angle: But the line CD (as well as BD and AD) being perpendicular

[o] Constr.
[p] Def. 1. 7.
[q] Ax. 10. 1.
[r] 14. 1.

pendicular to DG [p], it is therefore in the same plane (CDAB) with BD and AD [t]; and consequently, as the angles BAD, CDA are both right-ones [p], it must be also parallel to BA [u].

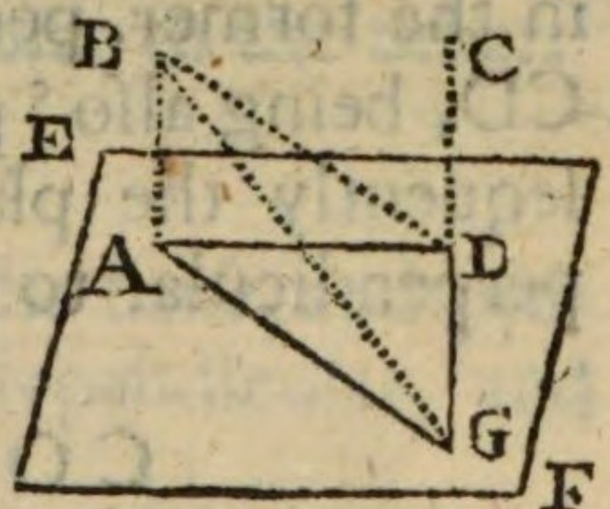
[p] Def. 1. 7.
[t] Cor. to 2. 7.
[u] 4. 1.

COROLLARY.

Hence it follows, that from the same given point, to one and the same plane, more than one perpendicular right-line cannot possibly be drawn: Because all perpendiculars to the same plane, are parallels; but lines drawn from the same point are not parallels.

THEOREM V.

If, of two parallel right-lines (AB, CD) *the one* (AB) *is perpendicular to any plane* (EF), *the other* (CD) *shall also be perpendicular to the same plane* (EF).

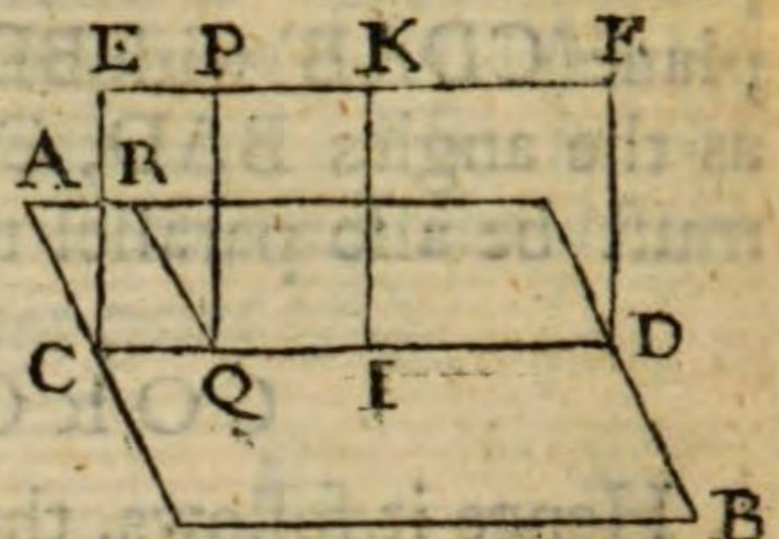


The construction of DG, AG, &c. being supposed the same here as in the preceding Theorem; it appears from thence, that ADG and BDG are both right-angles: And, because BD is in the plane of the proposed parallels BA, CD [w], the angle CDG is also a right-one [x], as is likewise the angle CDA [y]. Therefore CD is perpendicular to the plane ADG [z].

[w] Def. 6. 1.
[x] 2. 7.
[y] 4. 1.
[z] 2. 7. and Def. 1.

THEOREM VI.

If a right-line (PQ) *be perpendicular to a plane* (AB), *any plane* (ED) *passing by that line, will be perpendicular to the same plane* (AB).



In the plane ED, from any point K, draw KI perpendicular to the common ſection CD: Then, the angle DIK being a right-angle = DQP[y], IK will be parallel to PQ[z], and therefore perpendicular to the plane AB[a]. By the ſame inference, all other right-lines drawn in the plane ED, perpendicular to the common ſection CD, are alſo perpendicular to the plane AB. Therefore the plane ED itſelf is perpendicular to the plane AB[b].

[y] Def. 1. 7.
[z] 4. 1.
[a] 5. 7.
[b] Def. 2. 7.

COROLLARY I.

Hence it will appear, that the plane AB (according to the ſenſe of the definition) is perpendicular to the plane ED: For a right-line QR drawn in the former, perpendicular to the common ſection CD, being alſo [c] perpendicular to PQ, it [d] (and conſequently the plane AB in which it is) will be perpendicular to the plane ED [e].

[c] Hyp. and Def. 1. 7.
[d] 2. 7.
[e] 6. 7.

COROLLARY II.

Hence it alſo appears, that a line ſtanding at right-angles to one of two perpendicular planes at any point (I) in the common ſection (CD), muſt be in the other plane: For the line IK, in the plane ED, is perpendicular to the plane AB; beſides which line, another perpendicular to AB, from the ſame point I, cannot be drawn [f].

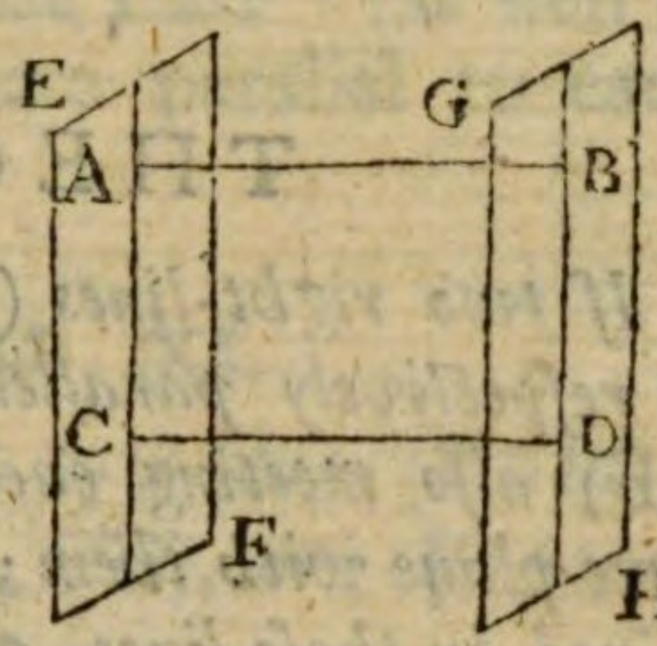
[f] Cor. to 4. 7.

THEOREM VII.

Planes (EF, GH) *to which one and the ſame right-line* (AB) *is perpendicular, are parallel to each other.*

From

From any point C, in the plane EF, let CD be drawn parallel to AB; which (as well as AB) will be perpendicular to both the planes [g]; and so the angles A, B, C, D (when AC and BD are joined) being all right-ones [h],



[g] 5. 7.

[h] Def. 1. 7.

the figure ABDC (whereof the sides AC, BD are in the same plane with the parallels AB, CD [i]) will therefore be a rectangular parallelogram [k]; and consequently CD = AB [l]. By the very same argument, all other perpendiculars, terminated by the two planes, are equal among themselves; *which was to be demonstrated* [m].

[i] Def. 6. 1.

[k] 4. 1. and Def. 24.

[l] 24. 1.

[m] Def. 3.

COROLLARY.

Hence, all right-lines perpendicular to one of two parallel planes, are also perpendicular to the other.

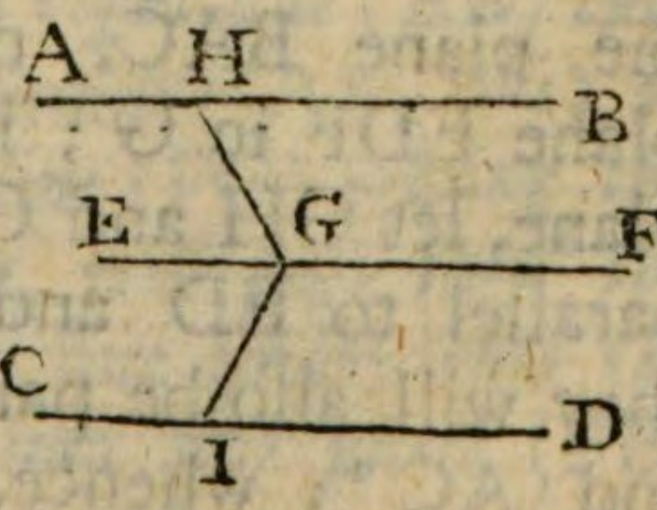
SCHOLIUM.

From the two last Theorems, the sense, and propriety of the two definitions of perpendicular, and parallel planes, appear manifest.

THEOREM VIII.

Right-lines (AB, CD) *parallel to one and the same right-line* (EF), *tho' not in the same plane with it, are also parallel to each other.*

Let GH and GI be drawn perpendicular to EF, in the planes AF and ED of the proposed parallels. Then [n] shall GF be perpendicular to the plane passing by HGI; and HB, ID will also be perpendicular to the same plane [o], and therefore parallel to each other [p].



[n] 2. 7.

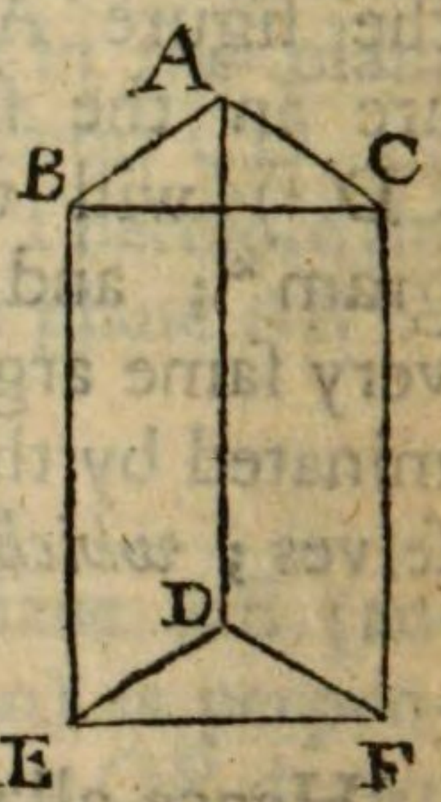
[o] 5. 7.

[p] 4. 7.

THEO-

THEOREM IX.

If two right-lines (AB, AC) *meeting each other, be reſpectively parallel to two other right-lines* (DE, DF) *alſo meeting each other, and not being in the ſame plane with them; the angles* (BAC, EDF) *contained by thoſe lines, will be equal.*

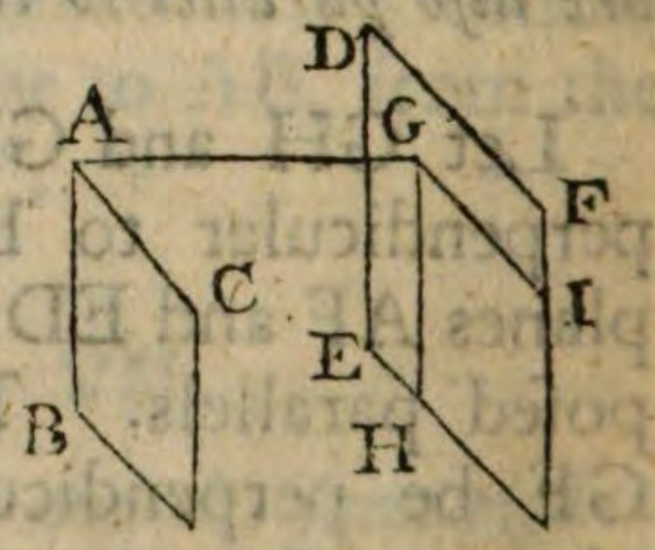


Take AB, AC, DE, DF all equal to each other, and let BE, AD, CF, BC, EF be drawn. Then, AB and ED, as well as AC and DF, being equal, and parallel [q], BE and CF will be both equal, and parallel, to AD [r], and therefore equal, and parallel, to each other [s]; whence BC is alſo equal to EF [r]; and ſo, the triangles ABC, DEF being mutually equilateral, the angles BAC, EDF are likewiſe equal [t].

[q] Hyp. and Conſtr.
[r] 26. 1.
[s] Ax. 1. and 8. 7.
[t] 14. 1.

THEOREM X.

If two right-lines (AB, AC) *meeting each other, be reſpectively parallel to two other right-lines* (DE, DF) *alſo meeting each other, and not being in the ſame plane with them, the planes* (BAC, EDF) *extended by thoſe lines, will be parallels.*



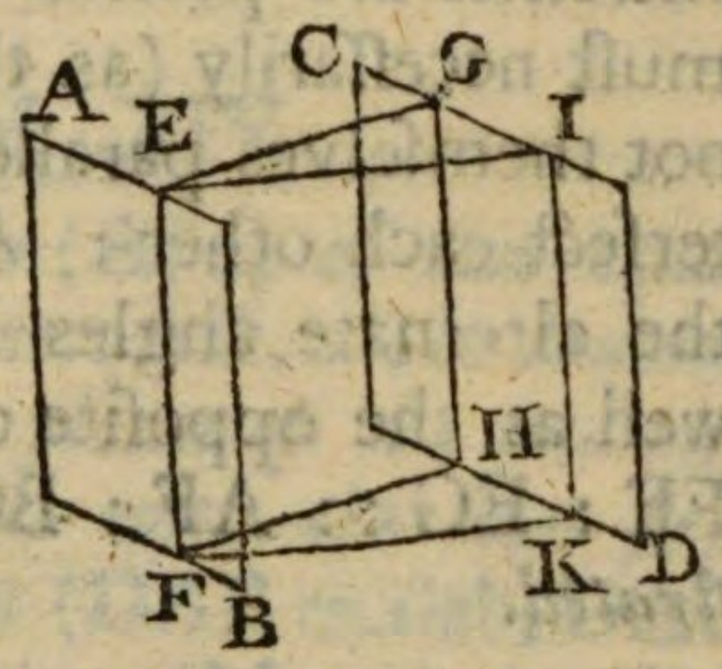
Let AG be perpendicular to the plane BAC, meeting the plane EDF in G; in which laſt plane, let GH and GI be drawn parallel to ED and DF; and they will alſo be parallel to AB and AC [u]; whence, ſeeing the angles GAB and GAC are both right [w], AGH and AGI muſt likewiſe be right-angles [x]; and ſo AG being

[u] 8. 7.
[w] Conſtr.
[x] 5. 1.

being perpendicular to the plane EDF [a] (as well as [a] 2. 7.
to BAC [b]), the two planes are parallel to each [b] Constr.
other [c]. [c] 7. 7.

THEOREM XI.

The sections (EF, GH) *made by a plane* (EFHG) *cutting two parallel planes* (AB, CD), *are also parallel, the one to the other.*



Let EG and FH be drawn parallel to each other, in the plane EFHG; also let EI, FK be perpendicular to the plane CD, and let IG, KH be joined: Then, EG being parallel to FH [d], and EI to FK [e], the angle GEI is = HFK [f]; but the angle EIG is also = FKH, being both right-angles [g]; and EI is = FK [h]: Therefore EG will be equal [i] (as well as parallel) to FH; and consequently EF likewise parallel to GH [k].

[d] Constr.
[e] 4. 7.
[f] 9. 7.
[g] Def. 1. 7.
[h] Def. 3. 7.
[i] 15. 1.
[k] 26. 1.

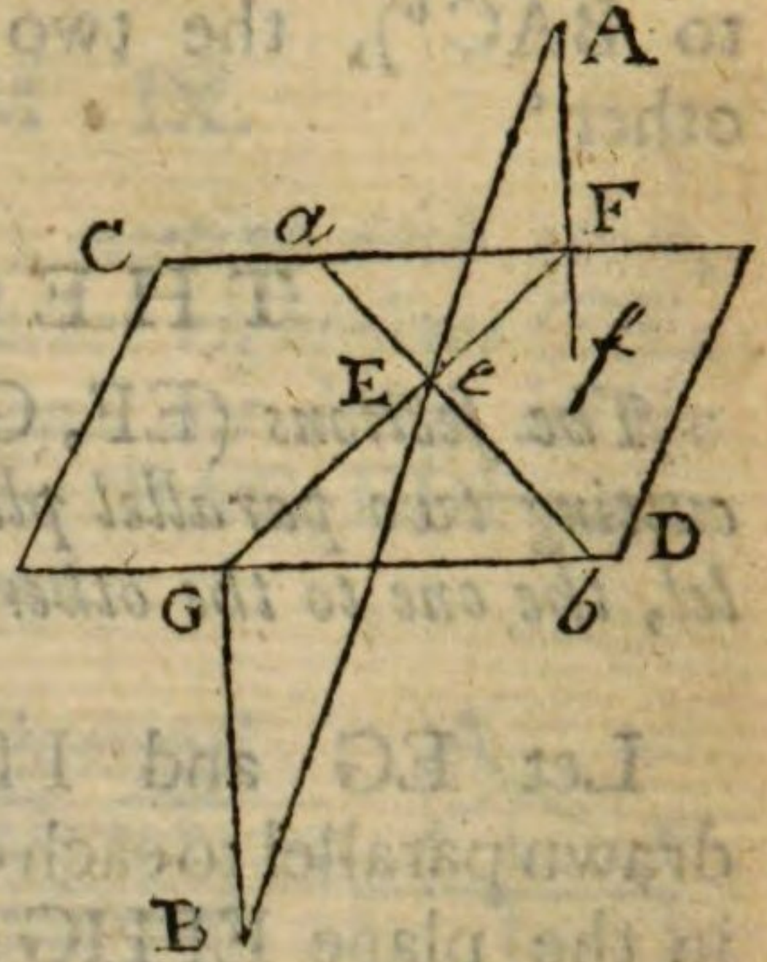
COROLLARY.

It appears from hence, that parallel lines, terminated by the same parallel planes, are equal to each other.

THEOREM XII.

If, from the two extremes of a right-line (AB) *cutting a plane* (CD), *two perpendiculars* (AF, BG) *be drawn to the plane; the right-line* (FG) *joining the points where they meet the plane, will pass thro' the point* (E) *in which the proposed line* (AB) *cuts the plane, so as to be divided by it into two parts* (FE, EG), *having the same ratio to each other as those two perpendiculars* (AF, BG).

For,



For, if AF be produced to *f*, the lines A*f* and BG (which are both perpendicular to the plane CD[l]) will be parallel to each other[m]; therefore AB and FG being both in the same plane with these parallels, in which their extremes are posited[n], they must necessarily (as they are not themselves parallels) interfect each other: And so the alternate angles FAE, GBE being equal[o], as well as the oppofite ones FEA, GEA[p], thence will FE : EG :: AF : BG[q]; *which was to be demonstrated.*

[l] Hyp.
[m] 4. 7.
[n] Def. 6, 1.
[o] 7. 1.
[p] 3. 1.
[q] 14. 4.

COROLLARY.

Hence, if in the plane CD, the lines FC, GD be made parallel, the one to the other, and in them be taken F*a* = FA, and G*b* = GB; then will the line (*ab*) joining the points *a* and *b*, cut FG in the very same point in which it is cut by AB. For, if *e* be taken as the interfection of *ab* and FG, the triangles *a*F*e*, G*eb* will be equiangular[r]; whence F*e* : *e*G :: F*a* (FA) : [s]G*b* (GB) :: FE : EG[t]. Therefore, feeing FG is divided in one and the same ratio, both by *e* and E, these points muft neceffarily coincide[u].

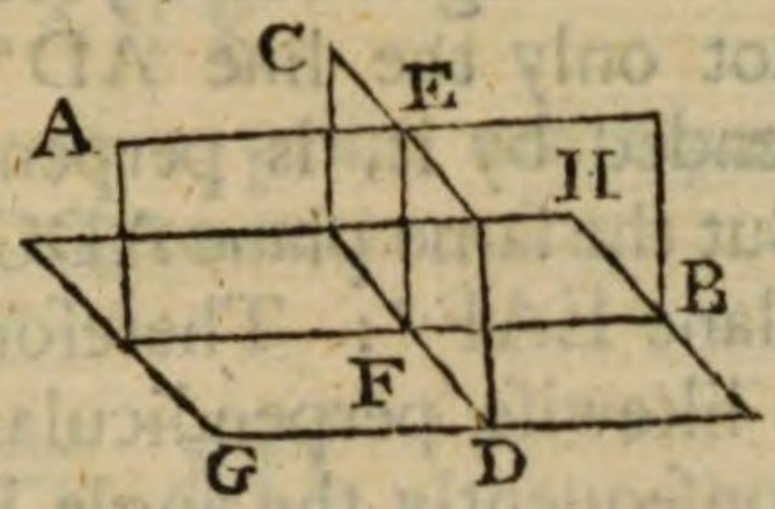
[r] 3. and 7. of 1.
[s] 14. 4.
[t] 12. 7.
[u] Ax. 2. and 5. of 4.

THEOREM XIII.

If two planes (AB, CD) *cutting each other, be both perpendicular to a third plane* (GH), *their common fection will alfo be perpendicular to the fame plane* (GH).

For,

For, from the extreme point F of the common section, let the right-line FE be erected perpendicular to the plane GH: which line being in both the planes AB, CD[l], it must necessarily be their common section. Therefore the common section is perpendicular to the plane GH[m].



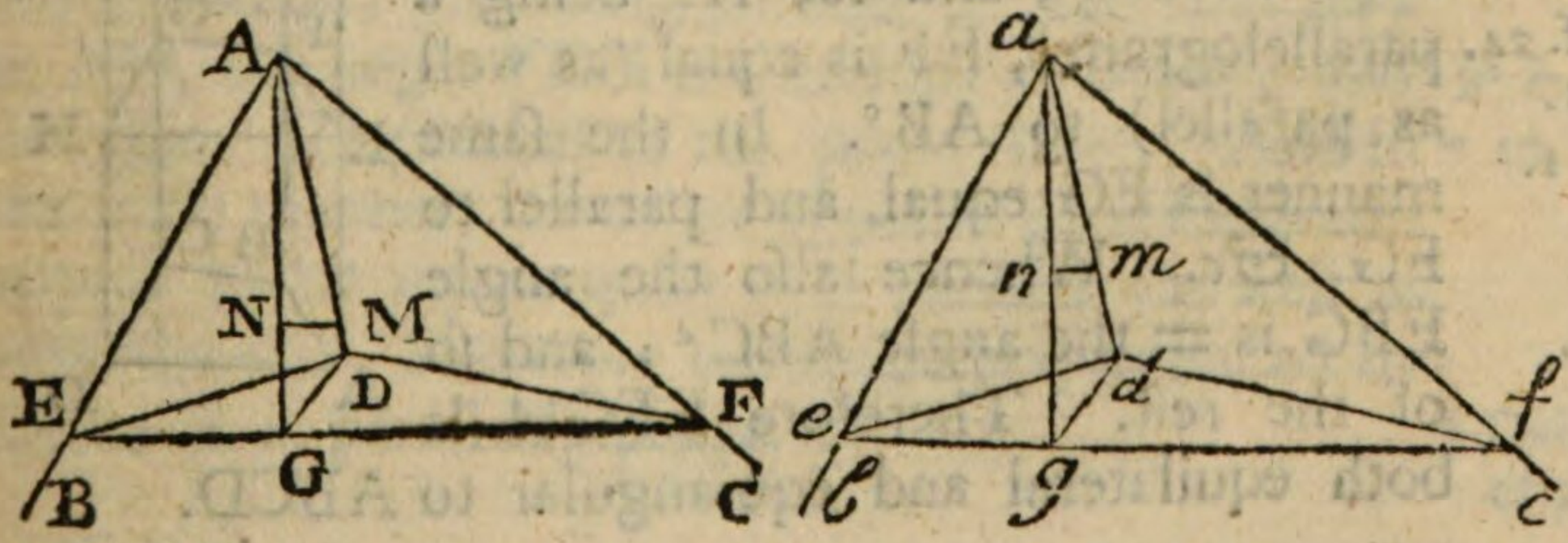
[l] Cor. 2. to 6. 7.

[m] Constr.

THEOREM XIV.

If, from the angular points (A, a) *of two equal angles* (BAC, bac), *two right-lines* (AD, ad) *be drawn, or elevated on high, above the planes of the said angles, so as to form equal angles with the lines first given, each to its correspondent* (DAB = dab, DAC = dac), *and if, from any points* (M, m) *in those elevated lines, perpendiculars* (MN, mn) *be let fall upon the planes* (BAC, bac) *of the first-mention'd angles; these perpendiculars will be, in proportion, as the parts* (AM, am) *of the elevated lines included between them and the angular points* (A, a) *first named.*

Make AD and *ad* equal to each other; and in the planes ADB, ADC, *adb*, *adc*, draw DE, DF, *de*, *df* perpendicular to AD and *ad*; and from their intersections with AB, AC, *ab*, *ac*, draw EF and *ef*, meeting AN and *an* (produced) in G and *g*, and let D, G, and *d*, *g* be joined.



The

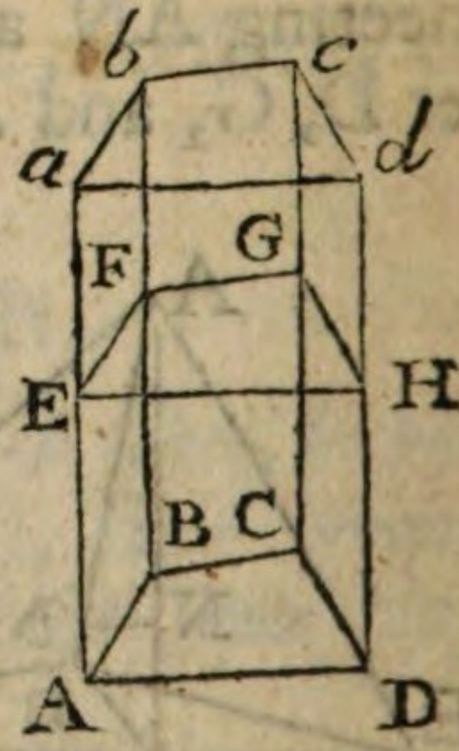
[n] Conſtr. The angles ADE, ADF being both right-ones [n];
[o] 2. 7. not only the line AD [o], but the plane ADG ex-
[p] 6. 7. tended by it, is perpendicular to the plane EDF [p].
But the ſame plane ADG is alſo perpendicular to the
plane EAF [p]: Therefore the common ſection EF
[q] 13. 7. is likewiſe perpendicular to the plane ADG [q]; and
[r] Def. 1. 7. conſequently the angle EGA a right-one [r]. By the
very ſame argument, *ega* is a right-angle. Now the
triangles ADE, *ade*; ADF, *adf* being equal in all
[s] Hyp. and 15. 1. reſpects [s], and the angle EAF = *eaf* [t], the triangles
AEF, *aef* are alſo equal and alike [u]; and ſo, the
[t] Hyp.
[u] Ax. 10. 1. angle AEG being = *aeg*, EGA = *ega*, and AE =
[w] 15. 1. *ae*, thence is AG = *ag* [w], and the angle DAG
[x] 16. 1. (MAN) = [x] *dag* (*man*), becauſe ADG, *adg* are both
[y] 14. 4. right-angles [o]. Therefore MN : *mn* : : AM : *am* [y].

COROLLARY.

Hence the two perpendiculars MN, *mn* ſubtend equal angles at the points (A, *a*) from whence the two elevated (or inclining) lines are drawn.

THEOREM XV.

If any ſolid (Ac), *having a rectilinear baſe* (ABCD), *whereof the planes* (Ab, Bc, Cd, Ad) *of the ſides are parallelograms, be cut by a plane parallel to the baſe, the ſection* (EFGH) *will be equal, and ſimilar to the baſe.*



For, the plane EFGH being paral-
[z] Hyp. lel to ABCD [z], EF is therefore pa-
[a] 11. 7. rallel to AB [a]; and ſo, AF being a
[b] Def. 24. of 1. parallelogram [b], EF is equal (as well as parallel) to AB [c]. In the ſame
[c] 24. 1. manner is FG equal, and parallel to FG, *&c.* Whence alſo the angle
[d] 9. 7. EFG is = the angle ABC [d]; and ſo of the reſt. Therefore EFGH is both equilateral and equiangular to ABCD.

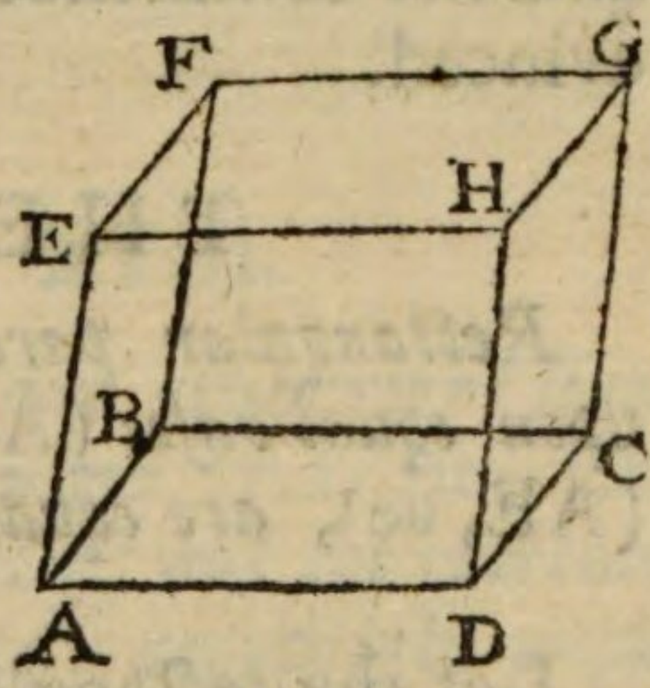
COROL-

COROLLARY.

Hence the oppoſite baſes of a priſm are equal and ſimilar (as well as parallel) to each other.

THEOREM XVI.

If, from one of the angular points (A) *of any parallelogram* (AC), *a right-line* (AE) *be elevated above the plane of the parallelogram, ſo as to make any angles* (EAB, EAD) *with the two contiguous ſides thereof, and there be alſo drawn, from the three remaining angular points, three other right-lines* (BF, CG, DH) *parallel, and equal to the former* (AE); *then, the extremes of thoſe lines being joined, I ſay, the figure* (AG) *thus deſcribed, will be a parallelepipedon.*



For AE, BF, CG, DH being all parallel to each other [e], AE and BF are in the ſame plane [f], as are alſo AE and DH, *&c.* Therefore, all theſe lines being [g] equal among themſelves, AF, AH, DG, and BG are parallelograms [h]; and ſo, EF being parallel to AB, parallel to DC, parallel to HG [i], EF and HG are in the ſame plane [f]; and EG is alſo a parallelogram [k], equilateral to its oppoſite AC: But EG is equiangular, and parallel (as well as equilateral) to its oppoſite AC; becauſe, EF being parallel to AB, and EH to AD, the angle FEH is therefore = BAD [l], and the plane EG parallel to the plane AC [m]. And in the ſame manner, the other oppoſite parallelograms appear to be equiangular and parallel (as well as equilateral). Therefore the ſolid AG, bounded by them, is a parallelepipedon [n].

[e] Hyp. and 8. 7.
[f] Def. 13. 1.
[g] Hyp.
[h] 26. 1.
[i] 8. 7.
[k] 25. 1.
[l] 9. 7.
[m] 10. 7.
[n] Def. 7. of 7.

COROL.

COROLLARY.

If the angle A of the parallelogram AC be a right-one, and AE be erected perpendicular to the plane AC; then will the parallelepipedon be a rectangular one: For, all the three contiguous planes AC, AF, AH being rectangular [o], their oppofites will be rectangular likewife [p]: And fo, the angles HGF, HGC being right-ones, HG will be perpendicular to the plane GB [q]; and confequently both the planes EG and DG likewife perpendicular to the plane BG [r]. And fo of the reft.

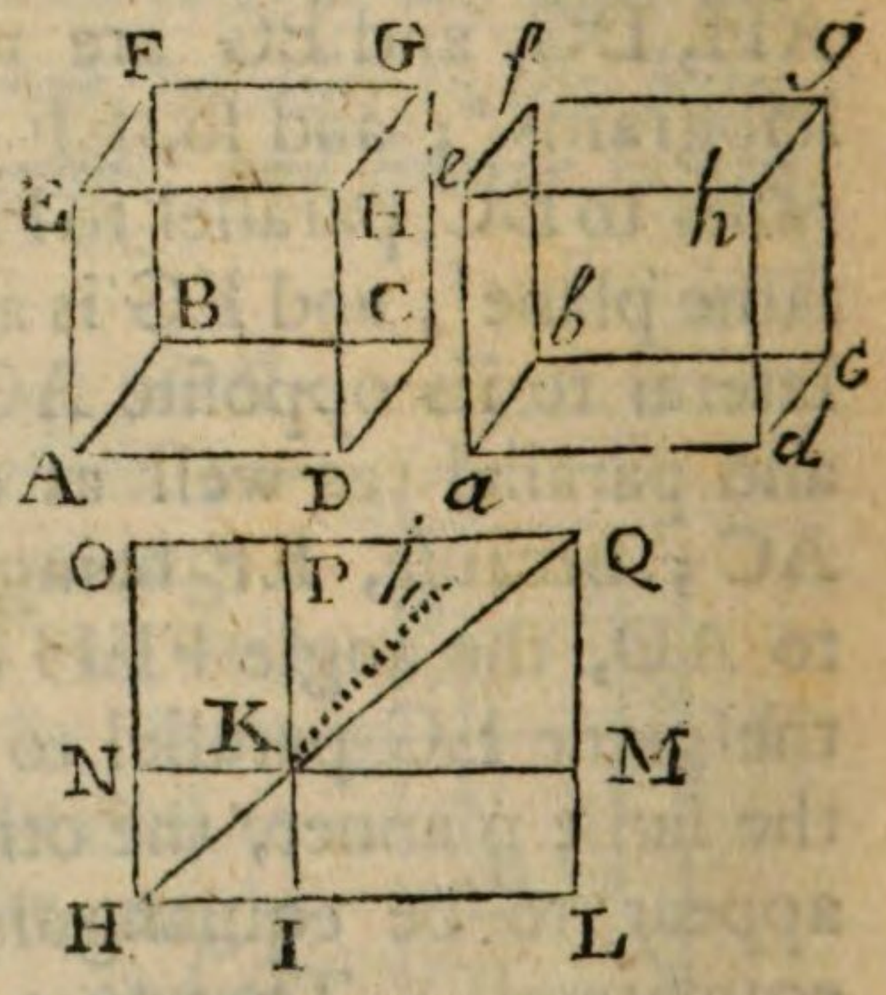
[o] Hyp. and 24. 1.
[p] 16. 7.
[q] 2. 7.
[r] 6. 7.

SCHOLIUM.

In this Theorem, a way to defcribe a parallelepipedon of any given dimenfions, is indicated; and the confiftence of the 7th and 9th definitions evinced.

THEOREM XVII.

Rectangular parallelepipedons (AG, ag) *ftanding upon equal bafes* (AC, ac), *and having equal altitudes* (AE, ae), *are equal.*



Let the rectangles OK and KL, equal and like to the bafes AC and *ac* of the two folids, be fo formed, that NK may be in the fame ftrait line with KM; then fhall PK be alfo in the fame ftrait line with KI [s]; and the figures NI, PM, OL, formed by producing the fides of the two rectangles, will likewife be rectangles [t].

[s] 2. 1.
[t] Cor. to 24. 1.

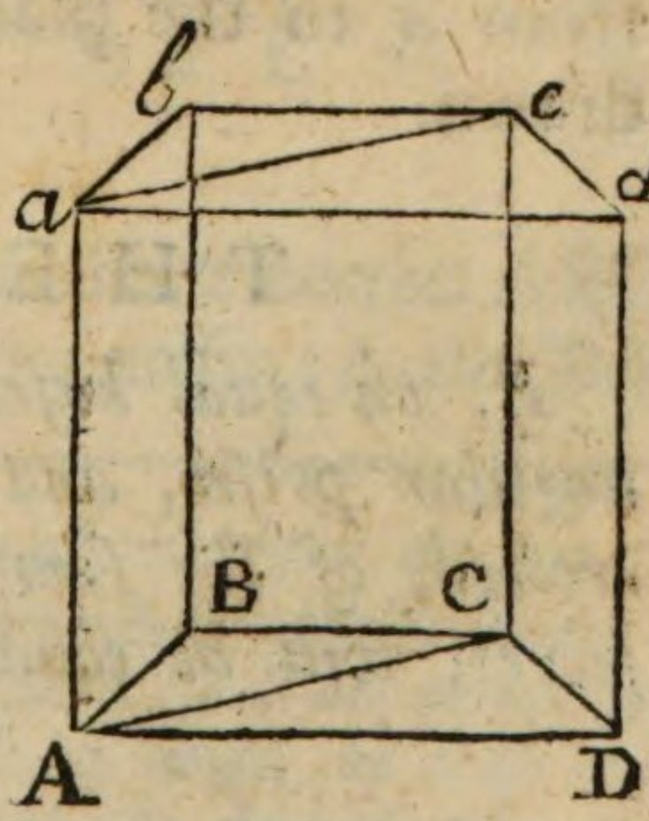
Now,

Now, HK and QK being drawn, the triangle NHK = IHK[u], PKQ = MKQ[u], and the rectangle OK = LK[w]; and consequently (by the addition of equals) OQKH = LQKH. Therefore, HKQ being a diagonal to the rectangle OL[x], dividing it into two equal, and like triangles OQH, LQH[u], if upon these, as bases, two upright prisms be conceived to be erected, of the same common altitude (K*b*) with the proposed solids, these prisms will also be equal[y]. But the former of these is compos'd[z] of three prisms, on the bases OPKN, NHK, KPQ; and the latter of three others, on KMLI, HIK, KMQ; whereof the second and third, in both ranks, are respectively equal[y]. Therefore the remaining two, on the bases OPKN, KMLI must also be[1] equal. But the former of these is = AG[y], and the latter = *ag*: Therefore, also, is AG = *ag*[a].

[u] 24. 1.
[w] Hyp.
[x] 24. 1. and Ax. 2.
[y] Ax. 1. 7.
[z] Cor. to 18. 7.
[1] Ax. 5.
[a] Ax. 1.

THEOREM XVIII.

If, at the angular points of any given right-lined figure (ABCD), *equal perpendiculars* (Aa, Bb, Cc, Dd) *be erected to the plane thereof, and the extremes of these* (a, b; b, c &c.) *be joined; an upright prism* (AabcdDCBA) *on the given base* (ABCD) *will thereby be formed.*



For, A*a*, B*b*, C*c*, D*d* being all equal[b], and parallel[c], it is evident that A*ab*B, B*bc*C, &c. are parallelograms[d]; and that the planes of these are all perpendicular to that of the base ABCD[e] (since A*a*, B*b*, C*c* are so, by construction). Moreover it will appear, that *abcd* is one plane figure, parallel to

[b] Constr.
[c] 4. 7.
[d] 26. 1.
[e] 6. 7.

ABCD; for, the lines *ac*, AC (when *a*, *c* and A, C are joined) being parallels [f] (as well as *ab*, AB; *ad*, AD), the plane *abc* is, therefore, parallel to ABC [g] (or ABCD); and *acd* is likewiſe parallel to ABCD. But *abc* and *acd* are in one plane; becauſe A*a* being perpendicular to both of them [h], and [i] conſequently to all the lines *ab*, *ac*, *ad*; theſe muſt neceſſarily be all in one [k] plane, parallel to ABCD; *which was to be demonſtrated* [l].

[f] 26. 1.
[g] 10. 7.
[h] Cor. to 7. 7.
[i] Def. 1. 7.
[k] Cor. to 2. 7.
[l] Def. 6.

COROLLARY.

It appears from hence, that, if upon all the parts ABC, ACD, into which any rectilineal figure ABCD is divided, upright priſms (A*abc*CB, A*acd*DC) of the ſame altitude be conſtituted; theſe priſms will form one priſm, on the (whole) given baſe ABCD; ſeeing that *abc* and *acd* form one continued plane ſuperficies *abcd* [m], parallel to ABCD.

[m] 18. 7.

SCHOLIUM.

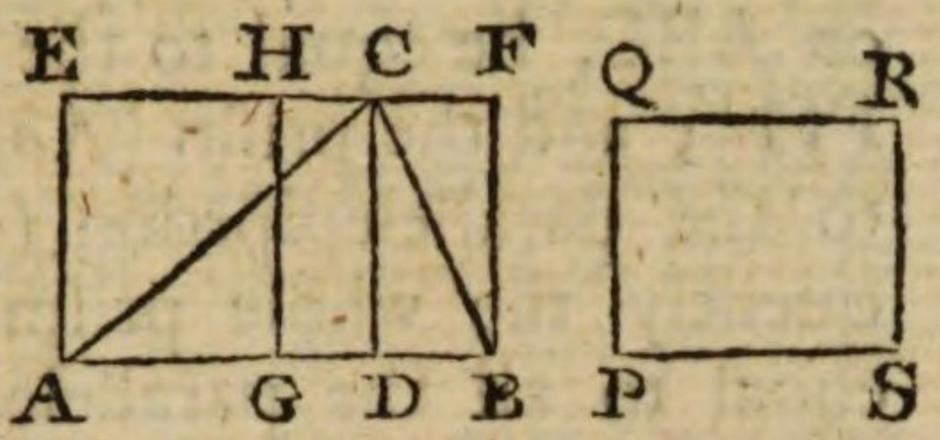
After the ſame way, a priſm, any how inclining on the given baſe ABCD, may be conſtructed; by giving to A*a* the propoſed inclination, and then drawing B*b*, C*c*, D*d* parallel, and equal thereto. For A*ab*B, B*bc*C, *&c.* will *(ſtill)* be parallelograms [f]: And, that *abcd* is one plane, parallel to ABCD, will alſo appear *(in the ſame manner)*; if a perpendicular from *a* to the plane ABCD, be conceived to be drawn.

[f] 34. 1.

THEOREM XIX.

If, on equal baſes (ABC, PQRS), *an upright triangular priſm, and a rectangular parallelepipedon be erected, of the ſame altitude; the two ſolids, themſelves, will be equal.*

Let

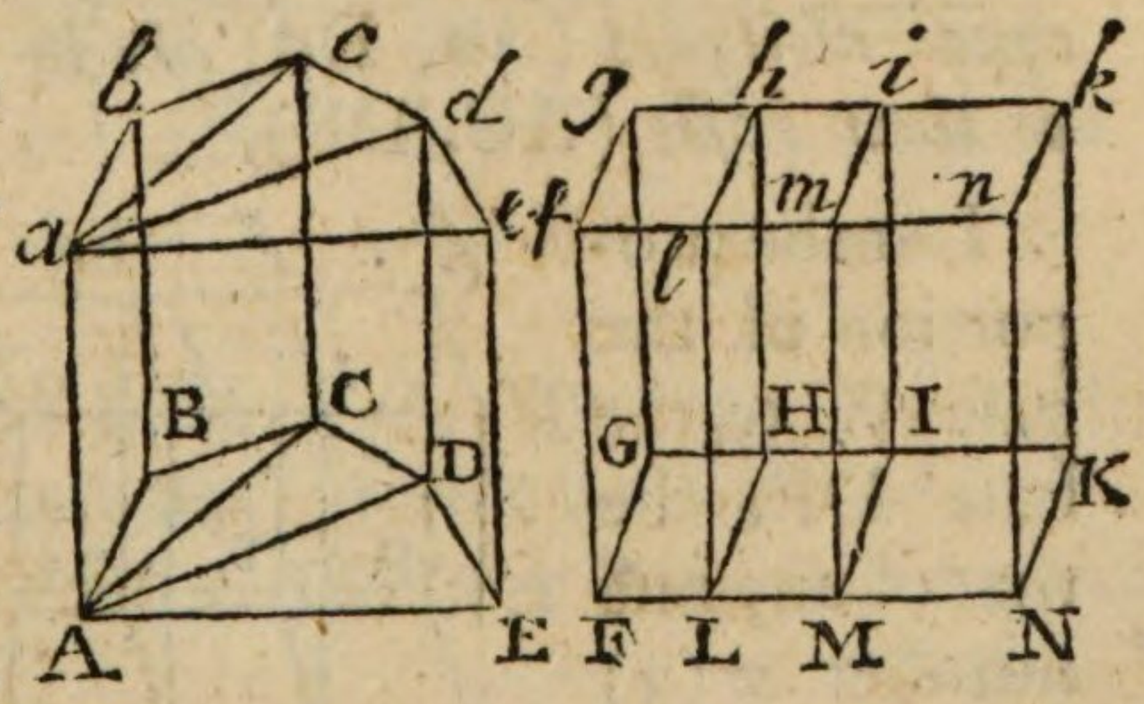


Let CD be perpendicular to AB, and let the rectangles ADCE, BDCF be completed; alſo let GH be drawn parallel to AE, biſecting AB in G; ſo ſhall AGHE = ½ABFE [b] = ACB [c] = PQRS [d]. Now, the two priſms on ADC and BDC, into which, *that* on ABC may be divided [e], will be reſpectively equal to two others, on the equal and ſimilar [f] baſes AEC and BFC [g]: and conſequently the priſm on ACB = half the priſm on AEFB [h] = half the two priſms on AGHE and BGHF [i] = the priſm on AGHE = the priſm on [k] PQRS *.

[b] 1. 2.
[c] Cor. to 2. 2.
[d] Hyp.
[e] Cor. to 18. 7.
[f] Ax. 24. 1.
[g] Ax. 1. 7.
[h] Ax. 4. 1.
[i] Ax. 3. 1.
[k] 17. 7.

THEOREM XX.

Every upright priſm (AaceA) *is equal to a rectangular parallelepipedon* (Fk) *of equal baſe, and altitude.*



Let AC, *ac*, AD, *ad* be drawn; and in the baſe FK of the parallelepipedon, let HL, IM, be drawn parallel to FG, in ſuch ſort that the rectangles FH, HM, IN may be reſpectively equal to the triangles ABC, ACD, ADE [k]: Then alſo ſhall the priſm (A*abc*CBA) [k] 6. 6.

---

* *In this Theorem, the repreſentations of the priſms are not deſcribed; becauſe a great multiplicity of lines, tends to produce confuſion in the mind of a learner; eſpecially, where ſolids are repreſented. The ſchemes, however, may be formed, at large, by thoſe who think proper to do it: But every tittle of the demonſtration will remain the ſame.*

on ABC, be equal to the parallelepipedon (F*b*) on
[1] 19. 7. FH[1]; and the prifm (A*acd*DCA) on ACD, equal to the parallelepipedon (L*i*) on LI[1]; and conſequently the whole prifm (A*ace*A) on ABCDE, equal to all the parallelepipedons on FK, which form one parallelepipedon (F*k*); becauſe L*l*, M*m* are
[m] 18. 7. and Cor. 2. to 6. 7. in the plane F*n* [m], and H*b*, I*i* in the plane G*k* [m].

COROLLARY.

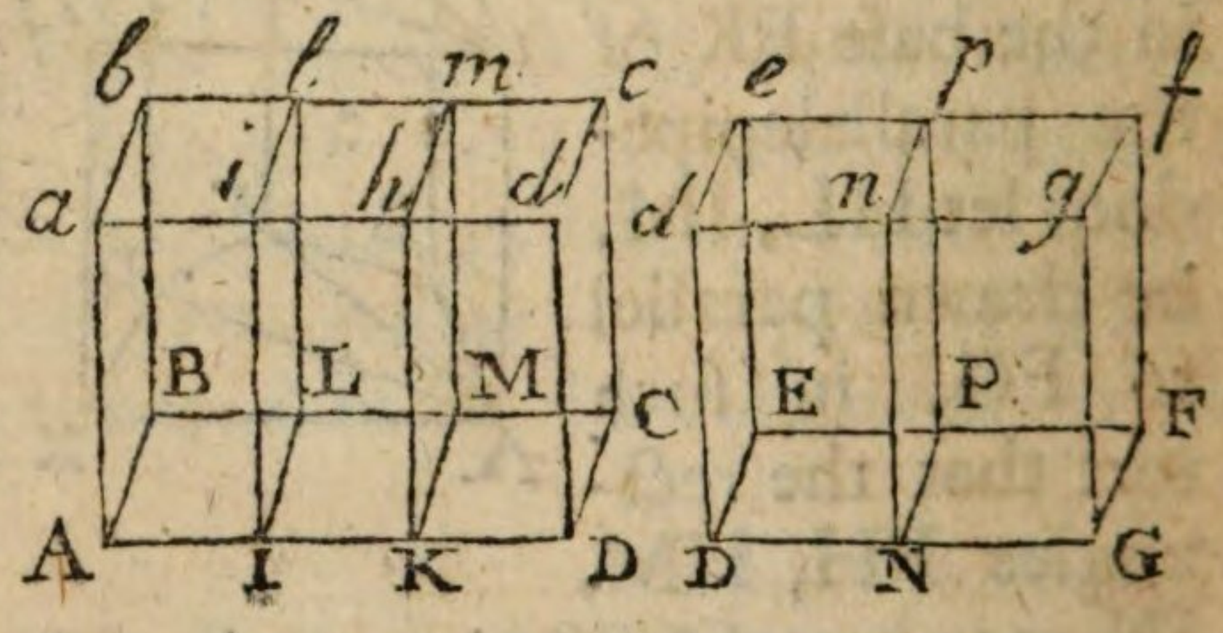
Hence all upright priſms, having equal baſes, and altitudes, are equal among themſelves.

SCHOLIUM.

In the very ſame manner, the aggregate of any number of priſms, of one common altitude, will appear to be equal to one ſingle priſm, or parallelepipedon, of the ſame altitude, whoſe baſe is equal to the ſum of all theirs.

THEOREM XXI.

Rectangular parallelepipedons (Ac, Df) *having equal altitudes* (Aa, Dd) *are in the ſame proportion as their baſes* (AC, DF).



Let the proportion of the baſe AC to the baſe DF, be that of any one number *m* (3) to any other number *n* (2).

Let AC be divided into *m* (3) equal parts (or rectangles) AL, IM, KC (by dividing AD into that number of equal parts [m], and drawing IL, KM parallel to AB) [n]: And let DF be divided, in like manner, into *n* (2) equal parts, or rectangles, DP, NF: Which parts, taken ſingly, will be equal, in magnitude, to thoſe of the former diviſion [o]; and ſo

[n] 11. 5.
[o] 9. 5. and 1. 2. Hyp. and Ax. 8. 4.

the

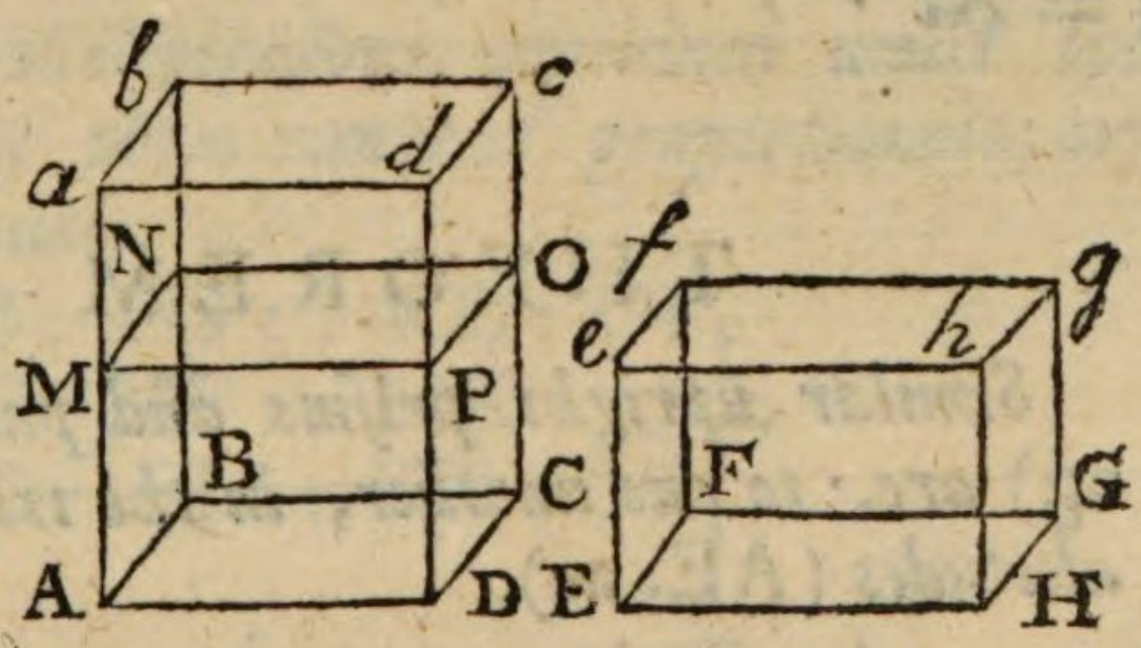
the parallelepipedons upon them (A*l*, I*m*, K*c*, D*p*, N*f*) will likewise be all equal[p]: Therefore the solid A*c* is in proportion to the solid D*f*, as the number of parts in A*c* to the number of equal parts in D*f*[r], or as the number of parts in AC to the number of equal parts in DF, that is, as AC to DF[r].—If the bases are supposed to be *incommensurable*, the solids will still be in the same ratio with them; as appears from the reasoning laid down in the *Scholium to Theor.* VII. *Book* IV.; which is equally applicable in this case.

[p] 17. 7.

[r] Ax. 8. 7.

THEOREM XXII.

Rectangular parallelepipedons (Ac, Eg) *standing upon equal bases* (AC, EG) *have the same ratio as their altitudes* (Aa, Ee).



Let AO be a parallelepipedon on the base AC, whereof the altitude AM is equal to that (E*e*) of the parallelepipedon E*g*: So shall the solid AO = the solid E*g*[s]. But (if A*b* and AN be considered as bases) it will be A*c* : AO (or E*g*) : : A*b* : AN[t] : : A*a* : AM[u] (or E*e*) : *which was to be proved.*

[s] 17. 7.

[t] 21. 7.

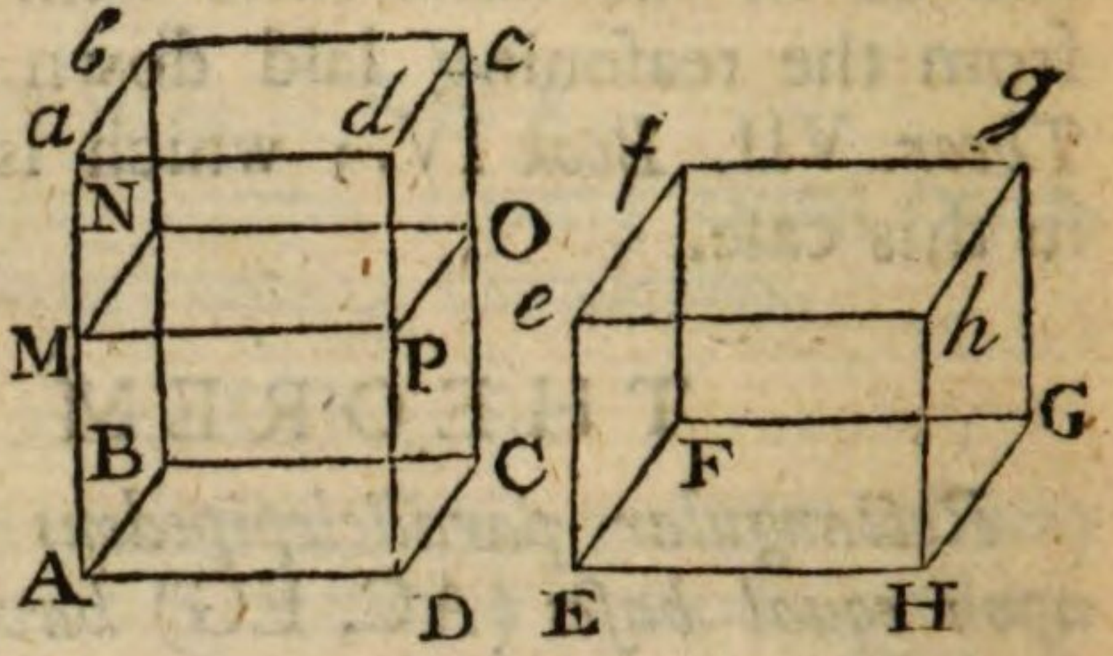
[u] 7. 4.

COROLLARY.

Hence, and from the preceding Theorem, it follows, that all upright prisms are, also, as the bases, when the altitudes are equal; and as the altitudes, when the bases are equal; all such solids being (*by Theor.* XX.) equal to rectangular parallelepipedons of equal base and altitude.

THEOREM XXIII.

Upright prisms and parallelepipedons (Ac, Eg) *which have their bases and altitudes reciprocally proportional* (AC : EG : : Ee : Aa), *are equal to each other.*



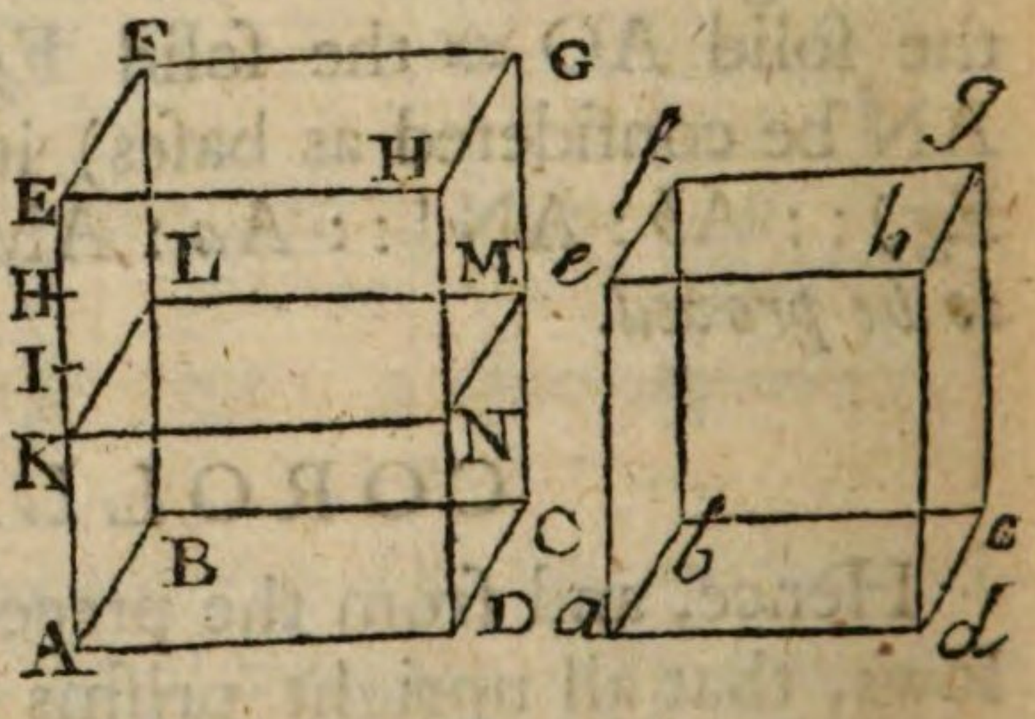
Let AO be a prism on the base AC, whereof the altitude AM is equal to that (E*e*) of the prism Eg.

Then AO : Eg :: AC : EG [w] : : E*e* (AM) : [x] A*a* : : AO : [y] A*c*; and consequently Eg = A*c* [z].

[w] 21. 7.
[x] Hyp.
[y] 22. 7.
[z] Ax. 4. of 4.

THEOREM XXIV.

Similar upright prisms and parallelepipedons (AG, ag) *are, to one another, in the triplicate ratio of their altitudes* (AE, ae).



Having made AH = *ae*, take AE, AH, AI, AK in continued proportion [a]; and let AM be a prism on the base AC, whereof the altitude is AK. Then (because of the similar planes) it will be AC : *ac* : : [b] AB^2 : *ab*^{2} : : [c] AE^2 : *ae*^{2} (AH^2) : : AH^2 : [c] AI^2 : : AH (*ae*) : AK [e]; and so, the bases and altitudes of the solids AM, *ag* being reciprocally proportional, the solids themselves are equal;

[a] 13. 5.
[b] 26. 4.
[c] Cor. to 11. 4.
[e] 27. 4.

equal[f]; and therefore, AG : *ag* : : AG : AM[g] : : AE : AK[h]: But the ratio of AE to AK is triplicate to that of AE to[i] AH (or *ae*). Therefore, *&c.*

[f] 23. 7.
[g] Ax 1. of 4.
[h] 22. 7.
[i] Def. 7. of 4.

COROLLARY I.

Hence, cubes are in the triplicate ratio of their ſides, or altitudes.

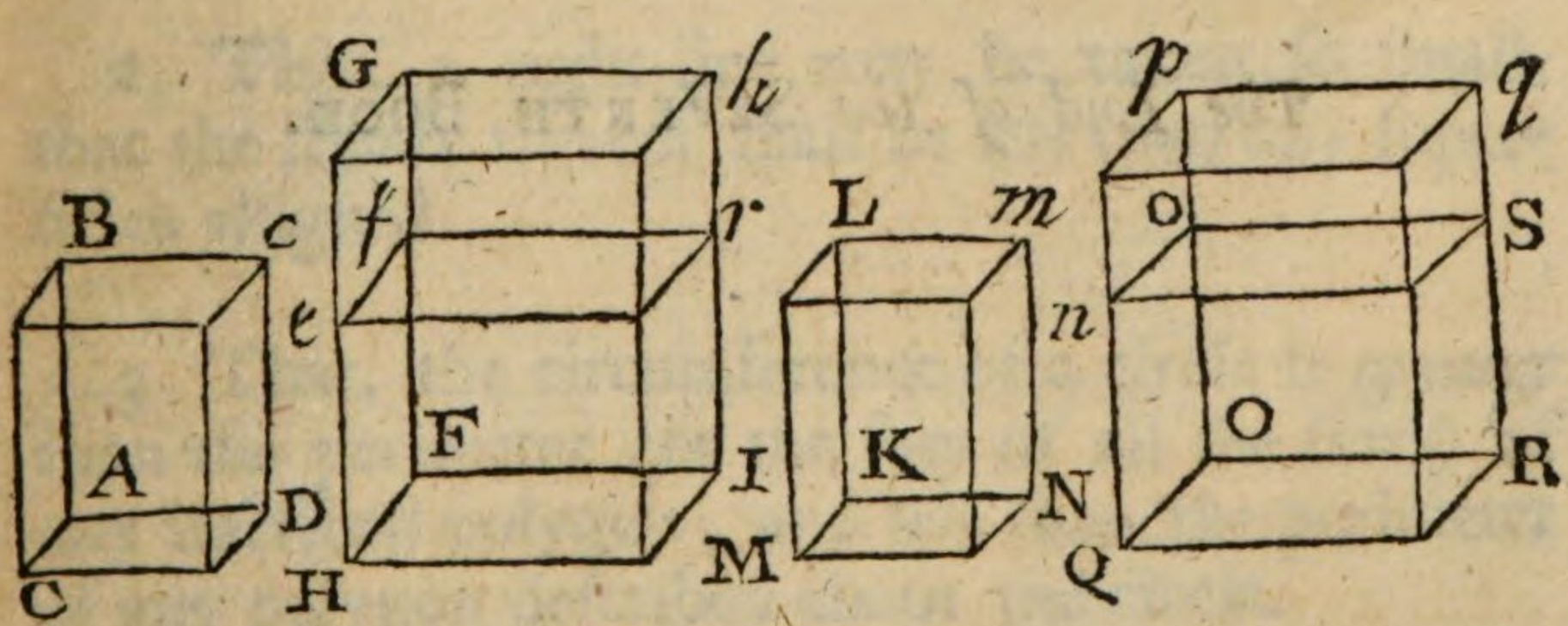
COROLLARY II.

Hence, alſo, all ſimilar upright-priſms, are to one another, as the cubes of their altitudes; ſince both priſms and cubes, are in the *ſame* triplicate ratio of the altitudes.

THEOREM XXV.

Rectangular parallelepipedons, contained under the correſponding lines of three ranks of proportionals, are themſelves proportionals.

I ſay, if { AB : FG : : KL : OP,
AC : HF : : MK : QO,
AD : FI : : KN : OR,



then ſhall the ſolid (Cc), *contained under the three firſt antecedents, be to that* (Hh) *contained under their three conſequents; as the ſolid* (Mm) *contained under the three other antecedents, is to that* (Qq) *contained under the three remaining conſequents.*

Let H*r* and Q*s* be parallelepipedons on the baſes HI and QR, of the ſame altitude with C*c* and M*m*, reſpectively.

21. 7. Then ſhall C*c* : H*r* : : baſe CD : baſe HI [k];

And M*m* : Q*s* : : baſe MN : baſe QR [k]. But the four baſes, becauſe of the proportionality of

11. 4. their ſides, are themſelves proportionals [l]: And ſo, by equality, the ratio of C*c* to H*r*, is the ſame as the ratio of M*m* to Q*s*. And the ratio of H*r* to H*b* is likewiſe the ſame as that of Q*s* to Q*q* (be-

[m] 22. 7. cauſe H*r* : H*b* : : F*f* (AB) : FG [m] : : KL (O*o*) :

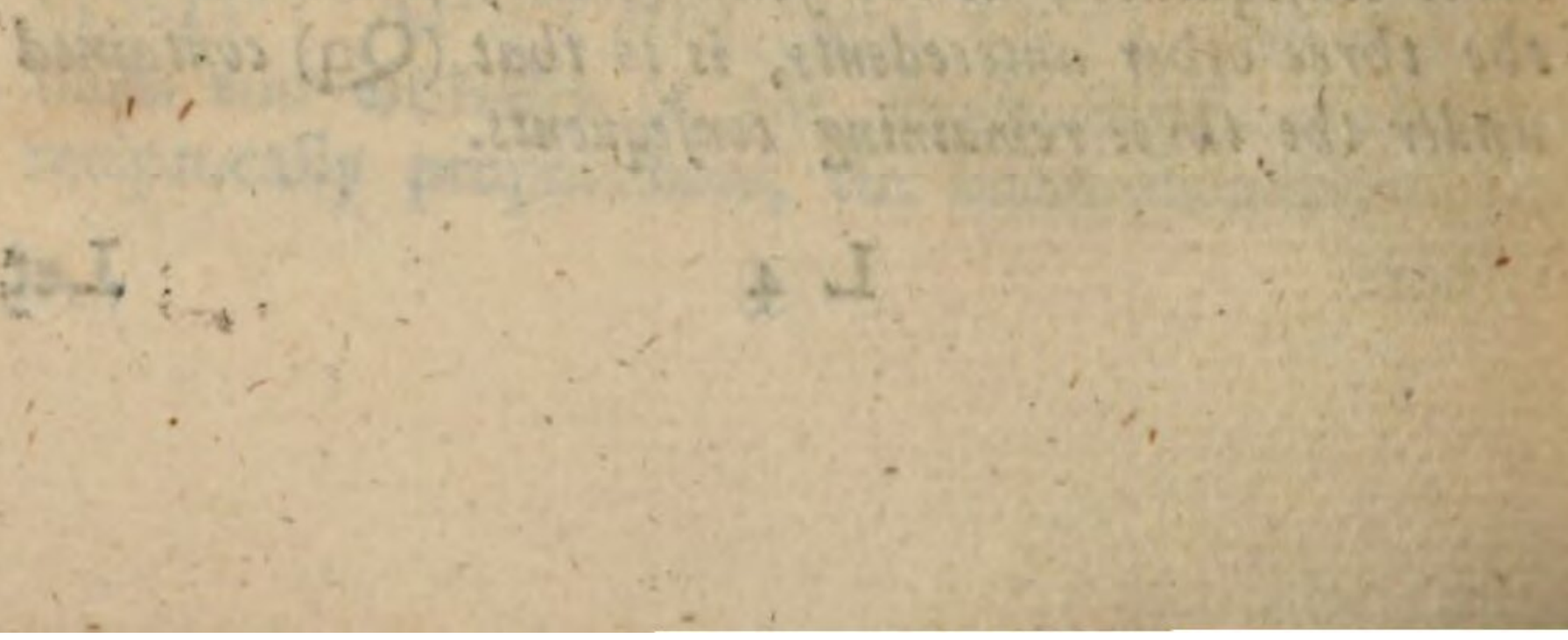
[n] Hyp. O*p* [n] : : Q*s* : Q*q* [m]). Therefore ſhall the ratio of C*c* to H*b*, be alſo the ſame, as the ratio of M*m*

[o] 5. 4. to Q*q* [o].

COROLLARY.

Hence, the cubes of four proportional lines are proportional.

The End of the SEVENTH BOOK.



ELEMENTS

OF

GEOMETRY.

BOOK VIII.

POSTULATES.

1. THAT, of any two unequal magnitudes, of the ſame kind, the leſs may be multiplied ſo often, till it exceed the greater.

2. That, a right-line may be taken ſo ſmall, that the ſquare thereof ſhall be leſs than any ſuperficies aſſigned.

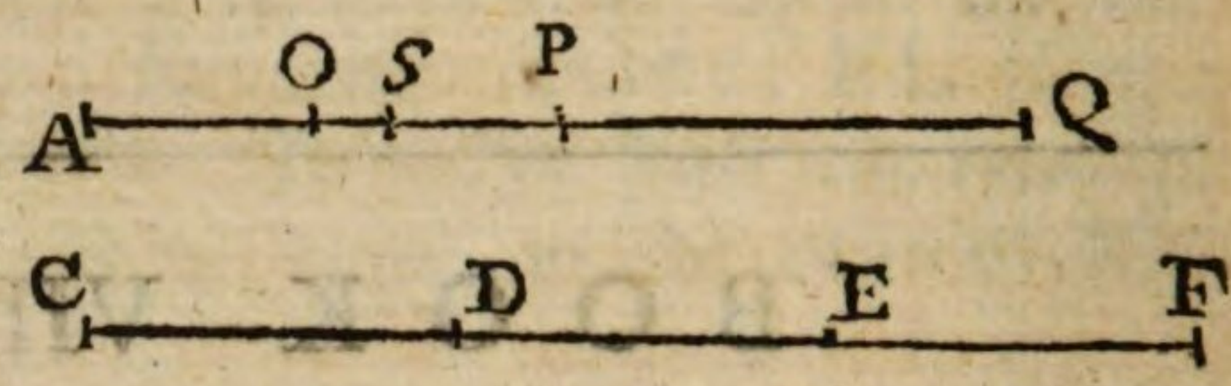
3. That, the circumference of a circle is greater than the perimeter (or the ſum of all the ſides) of any inſcribed polygon; and leſs than the perimeter of any polygon deſcribed about the circle.

What is required to be granted, in the ſecond of theſe three Poſtulates, might be effected and proved, in form, by means of the Firſt; but being itſelf more obvious (if poſſible) than even that, *it ſeemed unneceſſary to make it depend thereon.*

LEM.

LEMMA 1.

If from the greater (AQ) *of two unequal magnitudes* (AQ, CD) *there be taken the half* (PQ), *and from the remainder* (AP) *be again taken the half* (PO), *and ſo on, continually; there ſhall at length be left a magnitude, leſs than the leaſt* (CD) *of the two magnitudes firſt propounded.*



Take DE = CD, EF = CD, and let this be ſo often done, till the multiple CF exceed AQ [a]. Let the propoſed biſections of AQ, AP, *&c.* be continued till the parts PQ, OP, AO be equal in number to the parts EF, DE, CE. Now AP ($\frac{1}{2}$AQ [b]) ⊐ $\frac{1}{2}$CF [b] ⊐ CE. And, in the ſame manner, AO ($\frac{1}{2}$AP) ⊐ $\frac{1}{2}$CE (CD); *which was to be done.*

[a] Poſt. 1.

[b] Hyp.

SCHOLIUM.

When the magnitudes given (AQ, CD) are right-lines, a part, or meaſure (AS) of the one, leſs than the other, may be found at one operation [c]; by taking AS the ſame part of AQ, as CD is of CF [c]. For, the whole AQ being leſs than the whole CF [b], the part AS will alſo be leſs than the part CD [d].

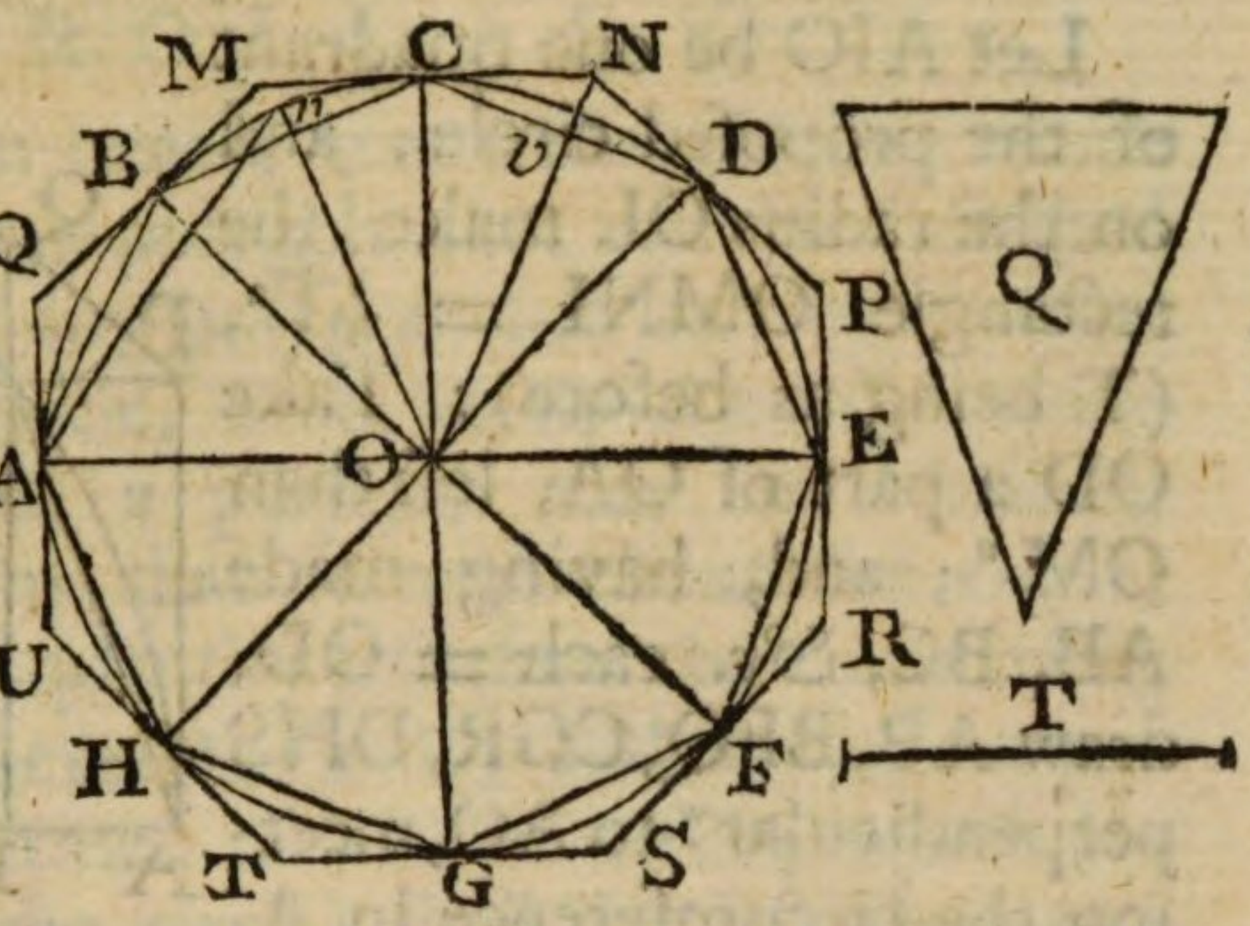
[c] 11. 5.

[d] Cor. 1. 4.

THEOREM I.

Two polygons may be formed, the one in, the other about a given circle, which ſhall differ leſs from each other (and conſequently from the circle itſelf) than by any aſſigned magnitude (Q) *however ſmall.*

Let



Let T be the ſide of a ſquare equal to, or leſs than Q [e]; and in the circle apply A*n* = T [f]: and, having drawn the two perpendicular diameters AE, CG, proceed by a continual biſection of the angles at the center, till you arrive at an angle AOB leſs than the angle A*On* ſubtended by A*n* [g]: Inſcribe the regular [h] polygon ABCDEFGH, by making the angles BOC, COD, &c. equal to AOB: and let another regular polygon QMNPRSTU, of the ſame number of ſides, [i] be deſcribed about the circle; which will exceed the inſcribed one by a magnitude leſs than Q.

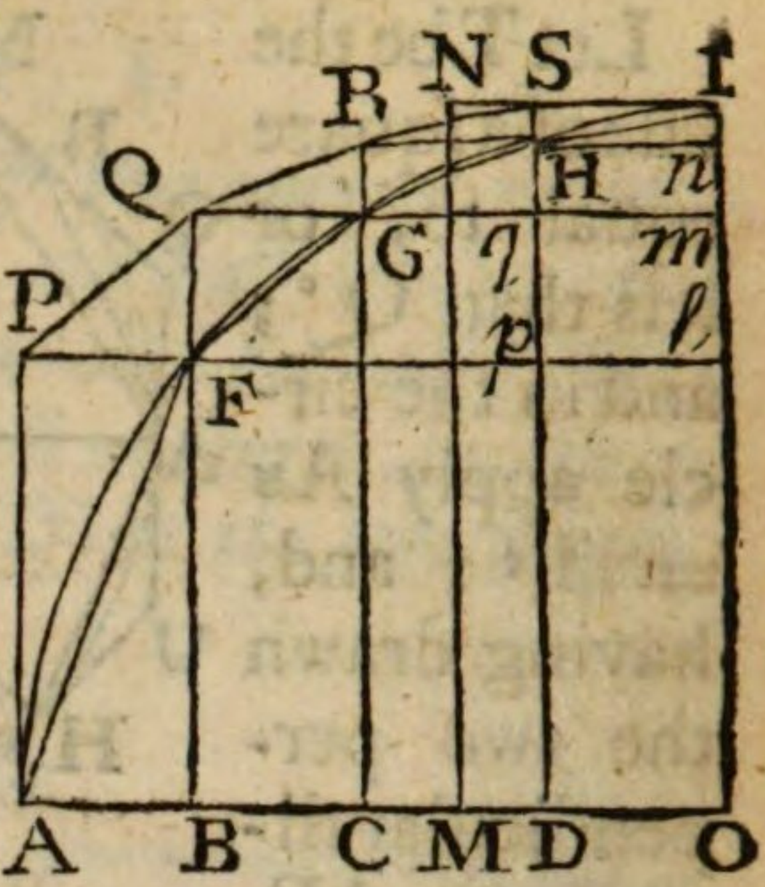
[e] Poſt. 2.
[f] 20. 5.
[g] 5. 5. and Lem. 1.
[h] Ax. 10. and 4. 1.
[i] 30. 5.

For, if to any angle N of the greater, ON be drawn, it will biſect the ſame [k], and will cut the ſide CD of the inſcribed polygon at right-angles [l] (in *v*): And ſo, the triangles OCN, *v*CN being equiangular [m], they (and conſequently their doubles OCNDO, CND) will be in proportion to each other, as [n] OC² to C*v*², or as [o] AE² to AB². And it is manifeſt, that the whole circumſcribing polygon (OCND + ODPE &c.) muſt be to its whole exceſs (CND + DPE &c.) above the inſcribed one, in the ſame [o] proportion of AE² to AB². But the firſt *antecedent* is leſs than the *ſecond*, or than a ſquare deſcribed about the circle [p]: Therefore the firſt *conſequent* (CND + DPE &c.) is alſo ⊐ [q] AB² ⊐ [r] A*n*² (T²) ⊐ [s] Q.

[k] 16. 1.
[l] Cor. to 12. 1.
[m] 19. 4.
[n] 24. 4.
[o] 1. 4.
[p] Ax. 2.
[q] 2. 4.
[r] 21. 1.
[s] Hyp.

Other -

Otherwiſe.



Let AIO be one quadrant of the propoſed circle; and
[t] 7. 6. on the radius OI, make [t] the rectangle OMNI $= \frac{1}{4}T^2$, (T being as before): Take OD a part of OA, leſs than
[u] Lem. OM [u]; and, having made AB, BC, *&c.* each = OD, draw AP, BFQ, CGR, DHS
[w] 3. 5. perpendicular [w] to AO, meeting the circumference in A,
[x] 9. 5. F, G, H; through which points, parallel to AO [x], draw PF*l*, QG*m*, RH*n*, meeting AP, BFQ, CGR, in P, Q, R: Join PQ, QR, RS, SI, as alſo AF, FG, GH, HI. Then will the two polygons OAFGHI and OAPQRSI (whereof one is leſs, and the other greater than the quadrant) differ leſs from each other, than by $\frac{1}{4}$ of the propoſed quantity Q.

For, that the former OAFGHI is leſs than the
[y] Ax. 2. quadrant, in which it is inſcribed, is manifeſt [y]: And, that the latter is greater than the quadrant, will alſo plainly appear; ſeeing two ſides AP, SI,
[z] 6. 3. only, touch the circumference [z]; all the reſt PQ, QR, RS, falling wholly above it, as being ſides of triangles PFQ, QGR, RHS formed out of the cir-
[a] Conſtr. and Ax. 2. cle [a]. Now the exceſs of the polygon OAPQRSI above OAFGHI, is compoſed of the triangle PAF ($= \frac{1}{2}$OD*pl*) and of all the parallelograms PFGQ,
[b] Ax. 3. QGHR [b], RHIS (for they are ſuch [c], becauſe PF
[c] 26. 1. (AB), QG (BC) are equal, as well as parallel [d]).
[d] Conſtr. and 26. 1. Therefore PFGQ being = *plmq* [e], QGHR = *qmn*H [e], *&c.* the ſaid exceſs will conſequently be
[e] Cor. 2. to 2. 2. = $\frac{1}{2}$OD*pl* + *lp*SI [b] ⊐ ODSI [f] ⊐ OMNI ($\frac{1}{4}T^2$)
[f] Ax. 2. ⊐ $\frac{1}{4}$Q [g]; *which was to be done.*—This laſt con-
[g] Hyp. ſtruction is equally applicable to other curvilineal figures; the former is peculiar to the circle.

COROL-

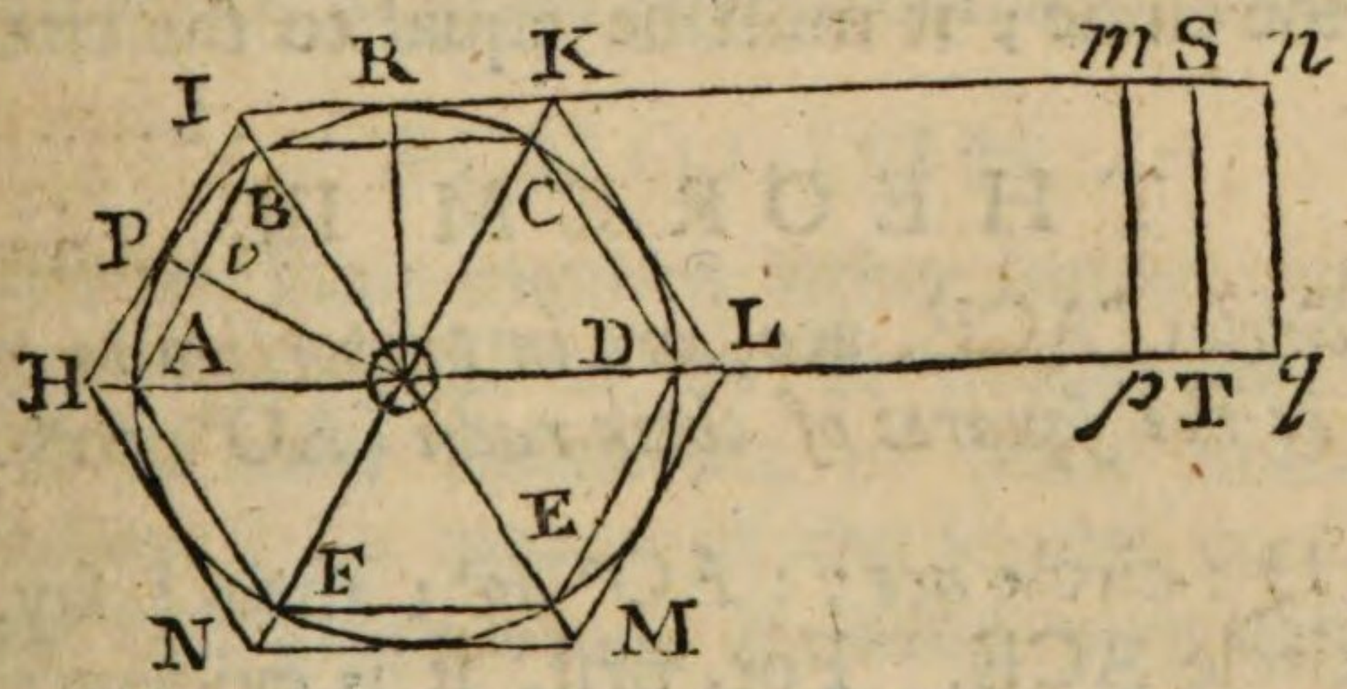
COROLLARY.

It follows from hence, that a magnitude, which is greater than any polygon that can be deſcribed in, and leſs than any polygon that can be formed about a given circle, muſt be equal to the circle itſelf: ſeeing that a polygon may be inſcribed, which (as well as that formed about the circle) ſhall exceed any quantity leſs than the circle itſelf, be the difference ever ſo ſmall; and becauſe a polygon may be formed about the circle, which (as well as that in the circle) ſhall be leſs than any quantity that exceeds the circle.

THEOREM II.

Every circle (ACE) *is equal to a rectangle* (ORST) *under the radius thereof* (OR) *and a right-line* (OT) *equal to half the circumference.*

It is evident, in the firſt place, that the propoſed rectangle ORST is greater than any polygon ABCDEF that can be deſcribed in the circle: For, drawing OA, OB, &c. and alſo O*v* perpendicular to AB; it is plain, that the triangle AOB ([h] O*v* × ½AB) will be leſs than [i] OA × ½AB (or OR × ½AB): And, in the ſame manner, BOC ⌐ OR × ½BC, &c. Conſequently, the whole polygon ABCDEF is leſs than [k] OR × ½AB + OR × ½BC, &c. that is [l], leſs than a rectangle (O*m*) under OR and O*p* = half the perimeter (AB + BC + CD &c.) But this rectangle (O*m*) is, itſelf, leſs than



[h] Cor. to 2. 2.
[i] 20. 1. and Ax. 2.
[k] Ax. 2. 1.
[l] 5. 2.

than OS, becauſe O*p* (half the perimeter of the
[m] Poſt. 3. polygon) is leſs than OT [m] (half the circumference of the circle). Conſequently the polygon ABCDEF is leſs than the rectangle OS.

But, ſecondly, it will appear, that the ſame rectangle ORST is leſs than any polygon HIKLMN that can be deſcribed about the circle: For, if OH, OI, &c. be joined, and the radius OP be drawn to the point of contact of HI; then will the
[n] Cor. to 2. 2. triangle HOI = [n] OP × ½HI (= OR × ½HI). In the very ſame manner IOK = OR × ½IK, &c. and therefore the whole polygon HIKLMN =
[o] Ax. 4. 1. [o] OR × ½HI + OR × ½IK, &c. = [p] a rectangle
[p] 5. 2. (O*n*) under OR and O*q* = half the perimeter (HI + IK + KL, &c.); which rectangle is, manifeſtly, greater than OS, ſince O*q* (= half the pe-
[q] Poſt. 3. rimeter of the polygon) is greater than OT [q].

Seeing, therefore, that the rectangle OS is greater than any polygon that can be deſcribed in the circle, and leſs than any polygon that can be deſcribed
[r] Cor. to 1. 8. about the circle; it muſt be equal to the circle [r].

THEOREM III.

All circles (ACE, ace) *are in proportion to one another, as the ſquares of their radii* (AO^2, ao^2).

Let Q : circle *ace* : : AO^2 ao^2; then I ſay, that Q = circle ACE. For, firſt, it is evident that Q is greater than any polygon ABCDEF that can be

deſcribed

deſcribed in the circle ACE: Becauſe, if another polygon *abcdef*, ſimilar thereto, be [s] deſcribed in the circle *ace*; then will polyg. ABCDEF : polyg. *abcdef* (: : [t] $AO^2 : ao^2$) : : Q : [u] circle *ace*; where the firſt *conſequent* (polyg. *abcdef*) being leſs than the *ſecond* (or, than the circle in which it is inſcribed [w]) it is manifeſt, that the firſt *antecedent* ABCDEF muſt alſo be leſs than the ſecond Q [x].

[s] 31. 5.
[t] Cor. to 31. 5.
[u] Hyp.
[w] Ax. 2.
[x] 2. 4.

In the ſame manner it will appear, that Q is leſs than any polygon HIKLMN that can poſſibly be deſcribed about the circle ACE: For, if about the other circle *ace*, a ſimilar polygon *hiklmn* be deſcribed [y]; then will HIKLMN : *hiklmn* (: : [z] $AO^2 : ao^2$) : : Q : [a] circle *ace*; where the firſt *conſequent* (*hiklmn*) being greater than the ſecond (*ace*) [b], the firſt *antecedent* HIKLMN muſt therefore be alſo greater than the [c] ſecond Q.

[y] 31. 5.
[z] Cor. to 31. 5.
[a] Hyp.
[b] Ax. 2.
[c] 2. 4.

Therefore, ſeeing that Q is greater than any polygon that can be deſcribed in the circle ACE, and leſs than any polygon that can be deſcribed about the circle; it muſt be equal to the circle [d].

[d] Cor. to 1. 8.

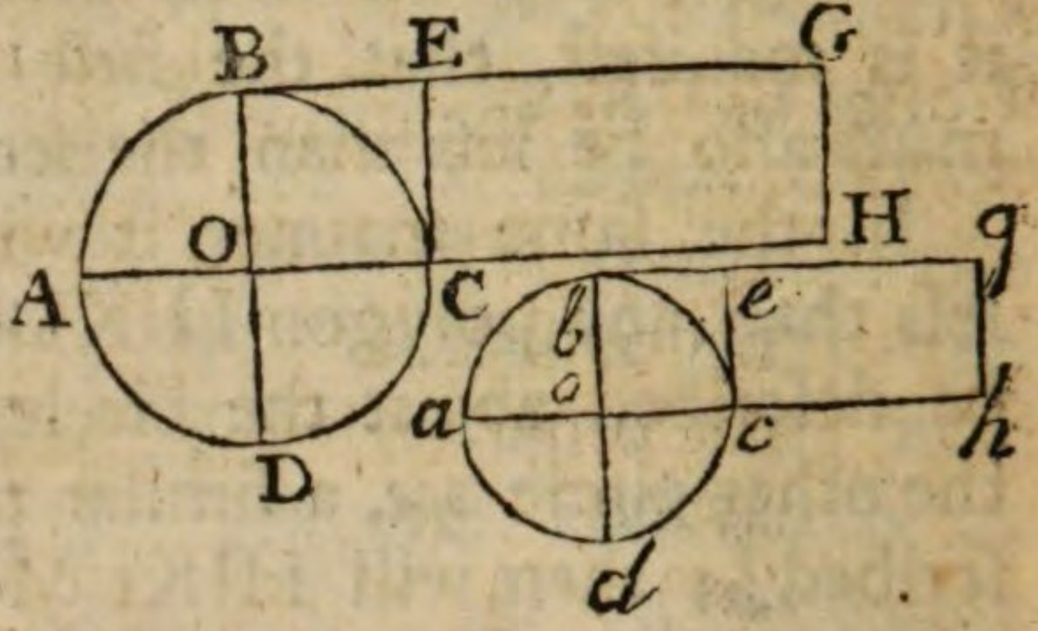
SCHOLIUM.

After the ſame manner, other ſimilar curvilineal figures are proved to be in proportion, as the ſquares of their diameters, or other homologous dimenſions; by means of the ſecond conſtruction of the firſt propoſition; it being very eaſy to demonſtrate, that the polygons formed from thence, whether both within, or both without two ſimilar figures, will themſelves be ſimilar.

THEO-

THEOREM IV.

The circumferences of all circles (ABCD, abcd) *are in the ſame proportion as their radii* (OB, ob).

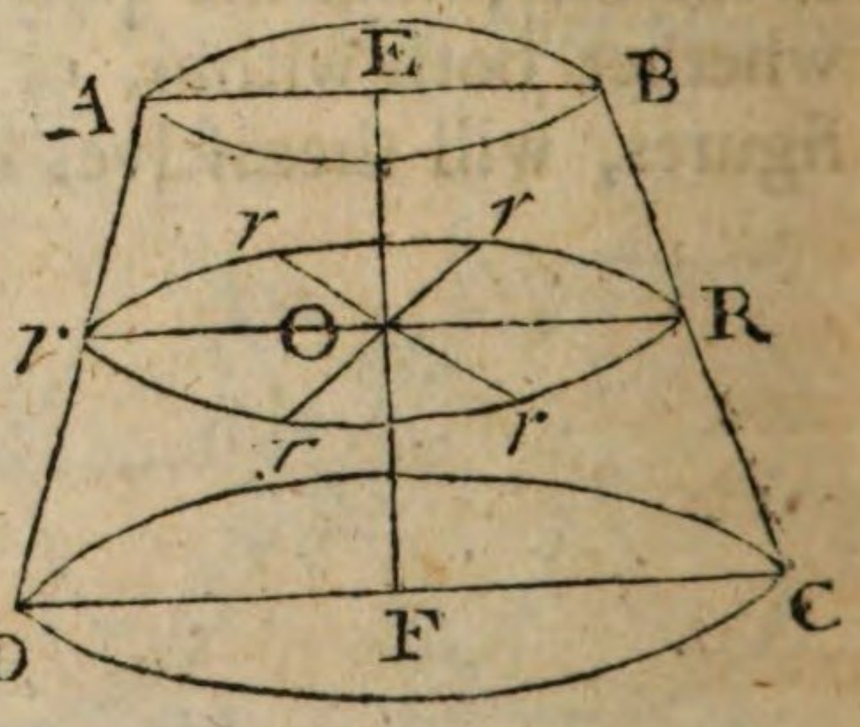


Let OE, *oe* be ſquares on the radii OB, *ob*; and let OG, *og* be two rectangles contained under the ſame radii and right-lines OH, *oh*, reſpectively equal to the ſemi-circumferences ABC, *abc*. Then, theſe rectangles being equal to the circles themſelves [d], it will therefore be, OE : OG : : *oe* : *og* [e]. And in this ſame ratio are [f] alſo the baſes OC, OH; *oc*, *oh*: whence *(by equality and alternation)* OC (OB) : *oc* (*ob*) : : OH : *oh* : : 2OH (circumf. ABCD [g]) : 2*oh* (circumf. *abcd*).

[d] 2. 8.
[e] 3. 8.
[f] 7. 4.
[g] Hyp.

LEMMA 2.

If a ſolid (AC) *generated by the revolution of any plane figure* (EBCF) *about a quieſcent axis* (EF), *be cut by a plane perpendicular to the axis; the ſection will be a circle, having its center in the point* (O) *where it meets the axis.*



For, from O, in the generating plane EBCF, draw OR perpendicular to the axis EF, meeting BC in R.

Then, ſince this line OR, during the whole revolution, every-where preſerves its perpendicu-

larity

larity to the axis EF, it is therefore always in the plane paſſing through O perpendicular to the ſaid axis [h]: and conſequently, as the length thereof alſo continues the ſame in every poſition, the line *Rrrrr* deſcribed, in that plane, by the extreme point R, by which the ſection is bounded, muſt be the circumference of a circle [i], whereof the point O is the center.

[h] Cor. to 2. 7.

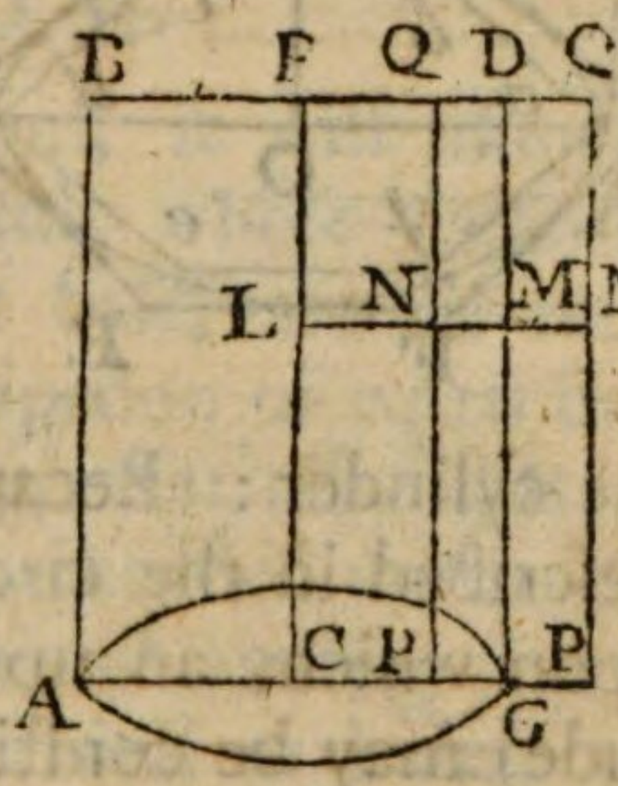
[i] Def. 33. 1.

COROLLARY.

Hence, not only the baſes of cylinders and cones, but all ſections parallel to them, are circles.

LEMMA 3.

A right-line (PQ) *ſtanding perpendicular to the plane of a cylinder's baſe (and not exceeding the axis* CF) *falls wholly within, or wholly without the cylinder, according as the point* (P) *on which it inſiſts, is ſituate within, or without the circumference of the baſe.*



From the center C, to the given point P, draw CP; take, in CF and PQ, any two equal diſtances CL, PN, and let LN be drawn, meeting the ſurface of the cylinder in M.

Becauſe CL and PN are parallel [k], and therefore both in the ſame plane [l], LN is parallel, and equal to CP [m]. Therefore, when CP is leſs than the radius CG, LN will be leſs than CG, or than its equal LM [n]; and ſo the point N muſt fall within the cylinder [o]. And the ſame is equally true with regard to any other point in the line PQ. But, when CP is greater than CG, LN will alſo be greater than HG (LM); and the point N will then fall out of the cylinder [o].

[k] 4. 7.

[l] Def. 14.

[m] 26. 1. 24. 1.

[n] Def. 12, 7. and

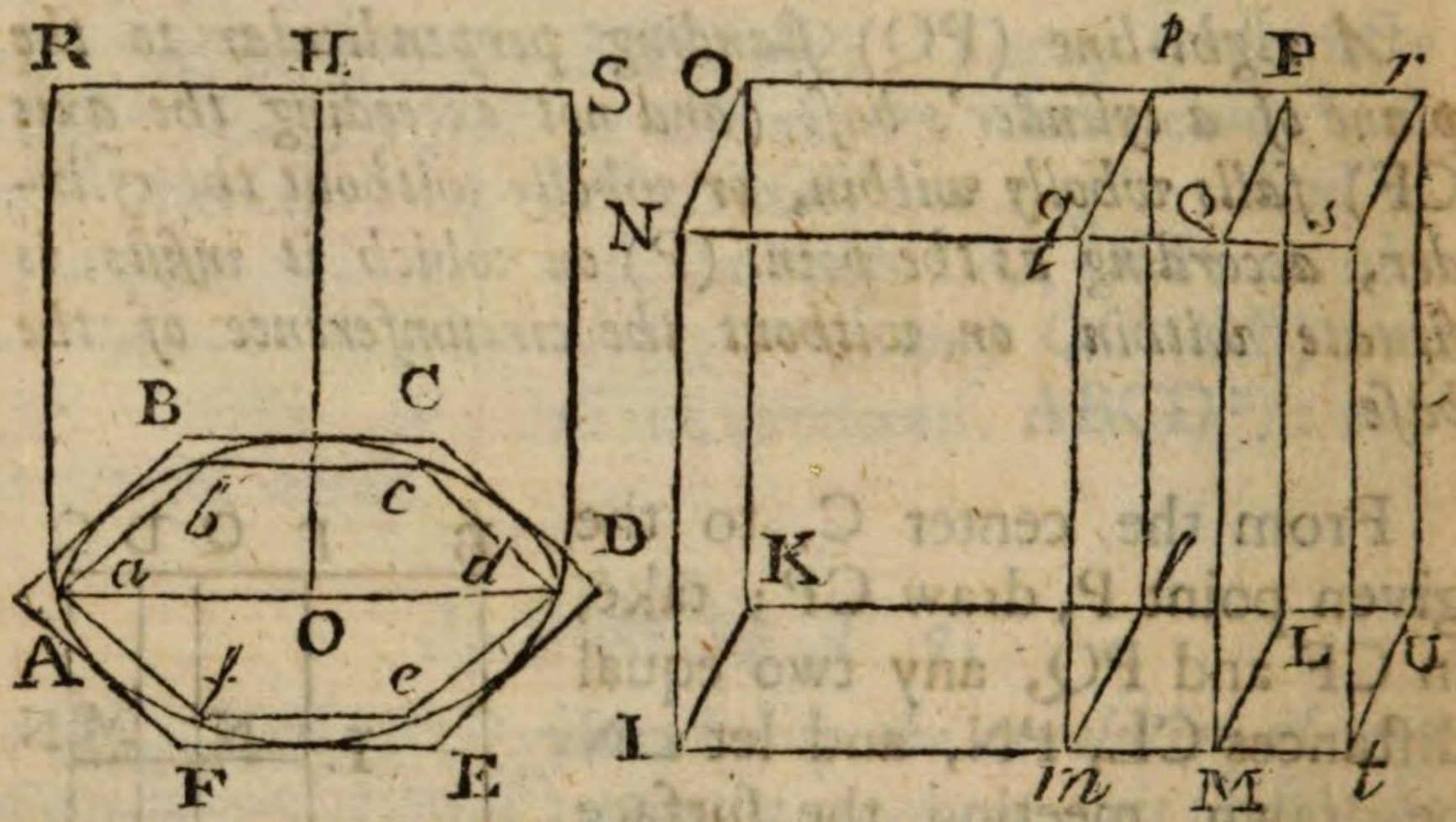
[o] Ax. 2. 1.

THEOREM V.

Every cylinder is equal to a rectangular parallelepipedon of equal base and altitude.

I say, if the base *ace* of the cylinder *a*S be equal to the base IKLM of the rectangular parallelepipedon IP, and the altitude OH of the former be also equal to the altitude KO of the latter; then the two solids will be equal.

For, first, it is evident, that the cylinder exceeds any parallelepipedon (I*p*), of the same given altitude, whose base I*klm* is less than the base (*ace*) of



the cylinder: Because a polygon (*abcdef*) may be described in the circle *ace*, that shall exceed IK*lm* [p] 1. 8.; upon which, an upright prism (of the given altitude) may be constituted [q] 18. 7.; which will be less than the cylinder, as being wholly contained therein; since (*by Lemma* 3.) all right-lines drawn perpendicular to the base, in the planes of the sides [r] Cor 2. to 6. 7. from any points in *ab*, *bc* &c. fall wholly within the cylinder, and consequently the planes themselves, in which they are. But this contained prism is greater than the parallelepipedon I*p* [s] 21. 7.: Therefore the cylinder itself must, necessarily, be greater than I*p* [t] Ax. 2.

In

In like manner it will appear, that the cylinder is leſs than any parallelepipedon Ir (of the ſame altitude) whoſe baſe IK*vt* exceeds that of the cylinder: For a polygon (ABCDEF) may be deſcribed about the circle *ace* that ſhall be leſs than IK*vt* [u]; upon which a priſm may be conſtituted [w], which, tho' leſs than Ir [x], will, nevertheleſs, exceed the cylinder [y].

[u] 1. 8.
[w] 18. 7.
[x] 21. 7.
[y] Ax. 2.

Therefore, ſeeing that the cylinder can neither be leſs, nor greater than IP; it muſt neceſſarily be equal to it.

COROLLARY.

Hence, whatever is demonſtrated in the 21ſt, 22d, and 23d Theorems of the preceding Book, with reſpect to the proportions of priſms, holds equally true in cylinders alſo; being equal to priſms of equal baſe and altitude [z].

[z] 20. 7. and 5. 8.

SCHOLIUM.

From the ſame demonſtration, it will likewiſe appear, that every regular ſolid, whoſe ſections, by planes perpendicular to the baſe, are all rectangles; is equal to a parallelepipedon of equal baſe and altitude; and conſequently, that all ſolids of this kind (which may be comprehended under the name of *Cylinderoids*) will be equal among themſelves, when their altitudes, as well as baſes, are equal.

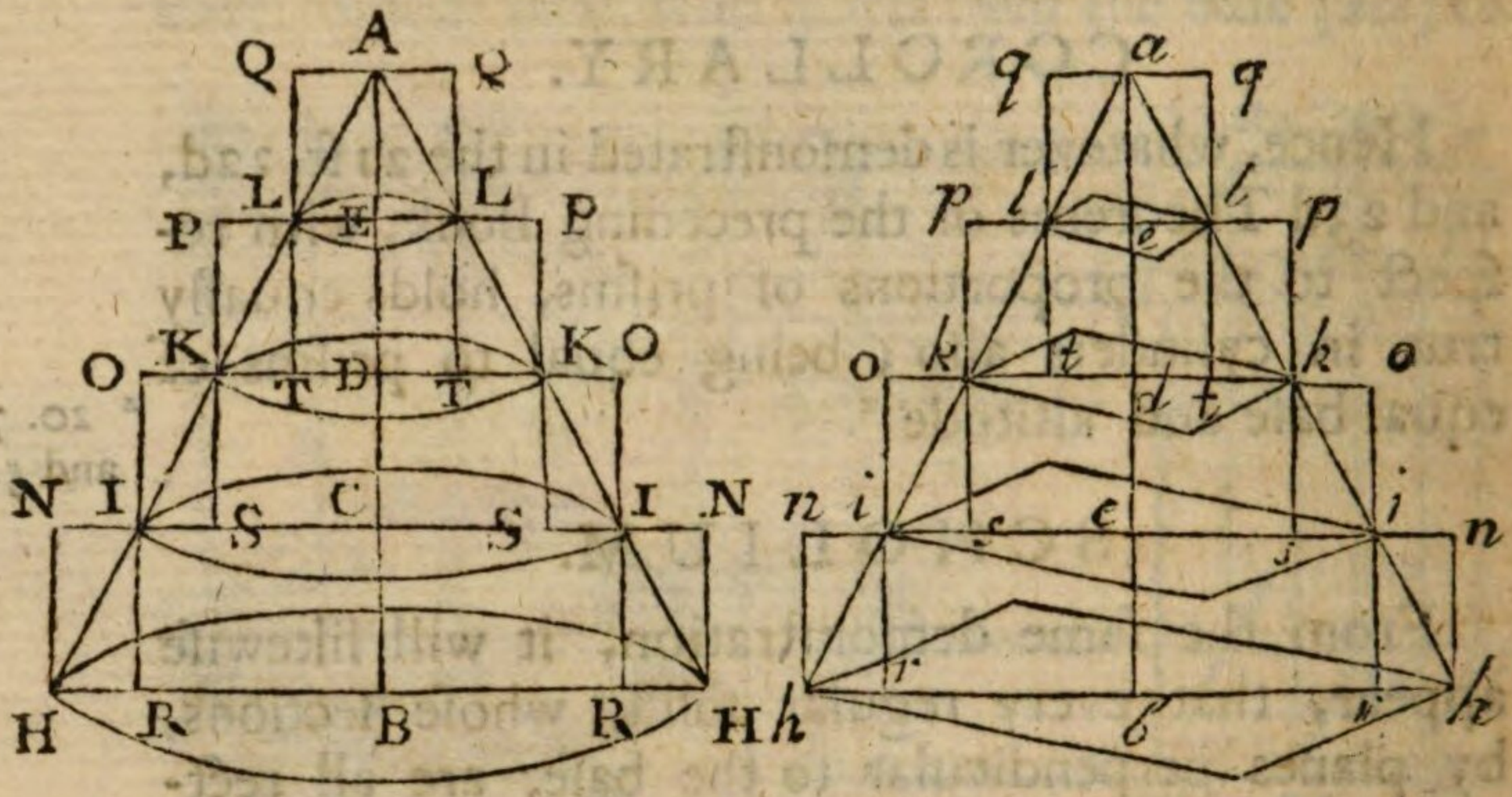
LEMMA 4.

If two ſolids (HAH, hah) *of the ſame altitude, have their ſections by planes parallel to the baſes, at all equal diſtances therefrom, equal to each other; it is propoſed to demonſtrate (under certain reſtrictions ſpecified hereafter) that the ſolids themſelves will be equal.*

Let II, KK &c. *ii*, *kk* &c. be ſections of the two ſolids by planes parallel to the baſes HH, *hh*, dividing the altitudes AB, *ab* into parts BC, CD &c. *bc*, *cd* &c. all mutually equal to each other. Then, every two correſponding ſections being equal [m] (HH = *hh*, II = *ii*, &c.) the upright ſolids HNNH, *hnnh*; IOOI, *iooi* &c. formed thereon, will alſo be, reſpectively, equal one to another [n], whether they be priſms, cylinders, or cylinderoids, that is, whether the ſections themſelves be right-lined figures, circles, or curvilineal figures of any other kind.

[m] Hyp.

[n] 20. 7. and Sch. 5. 8.



Now, if theſe ſections HH, II &c. be ſuppoſed to decreaſe, from the baſe upwards, ſo that the ſolids (HNNH, IOOI &c.) formed upon them may exceed the correſpondent parts (HIIH, IKKI &c.) of the given ſolid HAH; it is manifeſt, that the ſum of all the ſaid ſolids (HNNH + IOOI &c.) will likewiſe exceed the whole propoſed ſolid HAH°. But, if within HAH, on the ſame ſections (but on contrary ſides thereof) another ſeries of ſuch ſolids IRRI, KSSK &c. be formed; the ſum of all theſe will, manifeſtly, be leſs than the propoſed ſolid HAH, in which they are contained [p]: And it is alſo evident, that this laſt ſeries will be leſs than the former (HNNH + IOOI &c.) by the greateſt of theſe

° Ax. 2.

[p] Ax. 4.

theſe ſolids HNNH; becauſe (this one, alone, being excepted) to every other ſolid of the rank, an equal, in the contained rank, may be aſſigned, and *vice verſâ*: For [q] IRRI = IOOI, KSSK = KPPK, LTTL = LQQL.

[q] 20. 7. and Sch. 5. 8.

Now, ſince the altitude BC, of the ſolid HNNH whereby the contained, and containing ſeries' differ from each other, may be taken ſo ſmall a part of BA, that the ſolid itſelf ſhall be leſs than any aſſigned magnitude whatever [r]; it is manifeſt (from the reaſoning in Corol. to Theor. I.) that a magnitude, which is greater than any ſeries of ſolids (of the kind above ſpecified) that can be formed within the propoſed ſolid HAH, and leſs than any ſeries that can be formed about HAH, muſt be equal to HAH. But the ſolid *hah*, being greater than any ſeries of ſolids (*irri* + *kssk* &c.) contained therein [s], is therefore greater than any ſeries of ſolids IRRI + KSSK &c.) contained in HAH (*theſe*, being, reſpectively, equal to *thoſe*): And the ſame ſolid *hah*, being leſs than any ſeries of ſolids (*hnnh* + *iooi* &c.) formed about it, is alſo leſs than any ſeries of ſolids (HNNH + IOOI &c.) that can be formed about HAH. Therefore the ſolid *hah* is equal to HAH.

[r] Lem. 1. and 22. 7.

[s] Ax. 2.

In this demonſtration, the ſections are ſuppoſed to decreaſe, continually, from the baſes upwards; ſo as to have the ſides of the upright ſolids formed thereon, placed wholly without, or wholly within, the ſuperficies of the given ſolids HAH, *hah*: Which can only be the caſe, when all perpendiculars, from any points in the ſurface of either, to the plane of the baſe, fall within the limits of the baſe. If, however, the ſections be ſuppoſed to decreaſe to a certain diſtance, only, and then to increaſe again; the two ſolids will, ſtill, appear to be equal: Becauſe the parts of the one, terminated

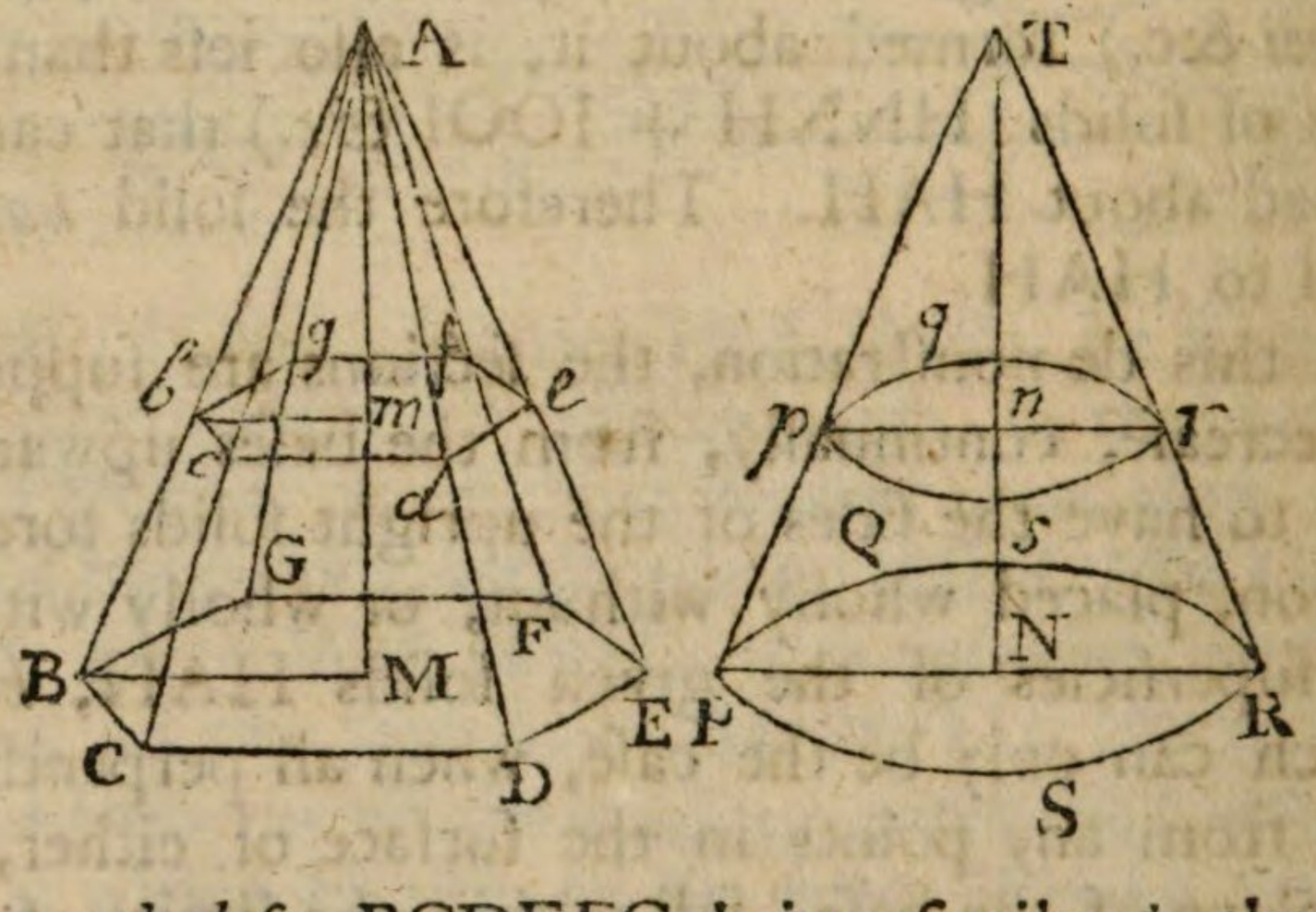
by ſuch limits of decreaſe, or increaſe, will (by the ſame demonſtration) be reſpectively equal to the correſpondent parts of the other. But, as no ſuch ſolids have a place in the *Elements of Geometry*, to ſay more about them here, would be improper.

LEMMA 5.

If pyramids and cones (ABCDEFG, TPQRS) *having equal altitudes* (AM, TN), *be cut by planes parallel to the baſes; the ſections* (bcdefg, pqrs), *at all equal altitudes* (Mm, Nn), *will be in the ſame proportion as the baſes.*

For, the plane *bcdefg* being parallel to BCDEFG, thence is *bc* parallel to BC [a], *bg* to BG, *&c.* and conſequently the angle *cbg* = CBG [b], *bcd* = BCD, *&c.* Alſo *bc* : BC (: : A*b* : AB [c]) : : *bg* : BG. And, in the ſame manner, the ſides about the other equal angles are proportional. Therefore, the two po-

[a] 11.7.
[b] 9. 7.
[c] 14. 4.



[d] Def. 14.
4.
[e] 26. 4.
[f] Cor. to 11. 4.
[g] Cor. to 7. 7.
[h] 4. 1.

lygons *bcdefg*, BCDEFG being ſimilar [d], they are in proportion [e], as *bc*² to BC², or as A*b*² to [f] AB², or, laſtly, as A*m*² to [f] AM²; becauſe (BM and *bm* being drawn) the angles AMB, A*mb* will be right-ones [g], and *bm*, therefore, parallel to BM [h].

But

But the ſection *pqrs* is *alſo* to the baſe PQRS in the *ſame* proportion of Am^2 (Tn^2) to AM^2 (TN^2); becauſe *pqrs*, PQRS, being [i] circles, they are as the ſquares of their radii *pn*, PN [k], and conſequently as Tn^2 to [l] TN^2. Therefore, ſeeing that the two ſections have both the ſame ratio to their reſpective baſes, the propoſition is manifeſt [m].

[i] Lem. 2.
[k] 3. 8.
[l] Cor. to 11. 4.
[m] 2. 4.

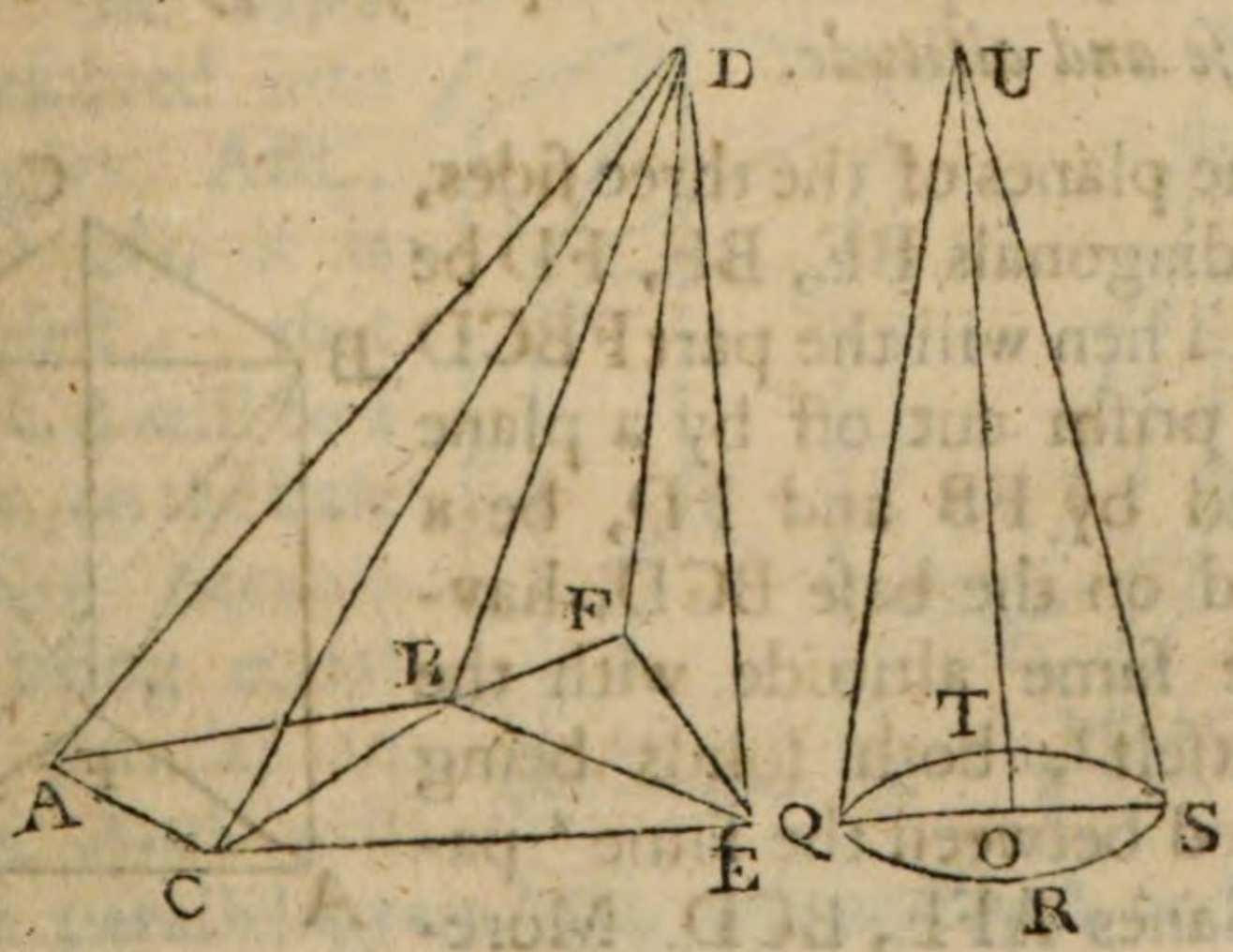
COROLLARY.

It appears from hence, that the ſection of any pyramid, by a plane parallel to the baſe, is ſimilar to the baſe.

THEOREM VI.

All pyramids of the ſame altitude, ſtanding upon equal triangular baſes, are equal among themſelves; and every ſuch pyramid (ABCD) *is equal to a cone* (QRSTU) *of equal baſe and altitude.*

Case I. If the perpendicular, let fall from the vertex D of the pyramid upon the plane of the baſe ABC, falls not out of the baſe, or beyond the limits of the triangle: Then it is manifeſt, *from Lemma* 4, ſeeing the ſections of the ſolids ABCD,

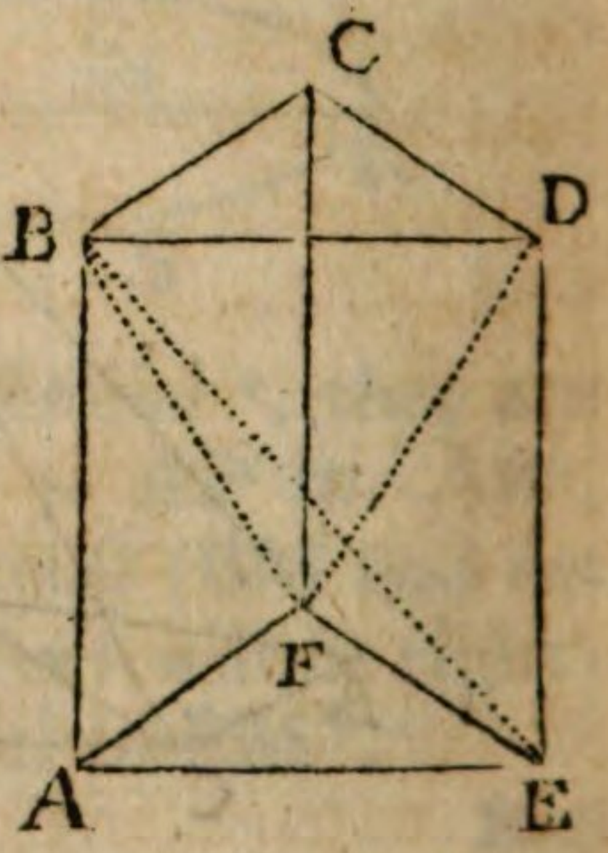


QRSTU, at all equal distances from the bases, will
[t] Lem. 5. be equal [t], that the solids themselves will likewise be equal.

CASE II. If the perpendicular (DE) from the vertex to the plane of the base, falls beyond the limits of the triangle: Then, to the point E where it meets the plane, let BE and CE be drawn; and on BE let a triangle EBF be described equal to
[u] Lem. 4. ABC (or QRST), and let F, D be joined. So shall the pyramid CBFED, standing on the base CBFE, be equal to the pyramid CABED, stand-
[w] Ax. 5. ing on the equal base CABE [w]; from each of which, let the common pyramid CBED be taken away; and there will then remain the pyramid
[x] Ax. 1. BFED = pyramid ABCD [w]: But the former of these is (*by Case* 1.) equal to the cone QRSTU; therefore it is evident, that the latter ABCD will also be equal to the [x] cone QRSTU; and, consequently, that all pyramids of the same altitude, standing on equal triangular bases, will be equal among themselves [x]; seeing every such pyramid is equal to a cone (QRSTU) of equal base and altitude.

THEOREM VII.

Every prism (ABCDEFA) *having a triangular base* (AFE) *is equal to the triple of a pyramid of the same base and altitude.*



In the planes of the three sides, let the diagonals BE, BF, FD be drawn. Then will the part FBCD of the prism cut off by a plane extended by FB and FD, be a pyramid on the base BCD, having the same altitude with the
[y] Def. 16. 7. prism itself [y], both solids being
[z] Def. 6. 7. contained between the same [z] parallel planes AFE, BCD. More-

over,

over, the remaining part FABDE of the prism, if a plane be extended by FB and FE, will be divided into the two pyramids FBAE, FDBE, which are equal to each other [a], as standing on the equal [b] triangular bases ABE, BDE. But the former of these pyramids FBAE, if B be now considered as the vertex thereof, will appear, also, to be equal to the first mention'd pyramid FBCD [a], the two bases AFE, BCD (as well as the altitudes) being equal [b]. Therefore, since the three triangular pyramids (FBCD, FABE, FBDE) into which the prism is resolved, are all equal to each other; the proposition is manifest.

[a] 6. 8.
[b] 24. 1.

COROLLARY.

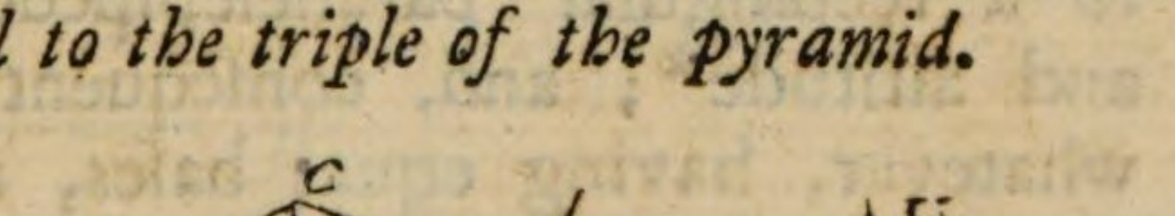
Hence, every prism having a triangular base, is equal to the triple of any pyramid of the same altitude, standing upon an equal triangular base [c].

[c] 6. 8. and Ax. 1.

THEOREM VIII.

If a prism (AbcE) *and a pyramid* (PQRSTU) *stand upon equal, and similar bases* (ABCDE, PQRST), *and have both the same altitude; the prism will be equal to the triple of the pyramid.*

If the bases be resolved into triangles, ABC, ACD &c. it is manifest, that B*bac*CA will be a prism, on the base ABC; because C*c* being equal and parallel to A*a*, A*ac*C will be a parallelogram [e] (as well as B*ba*A and B*bc*C [f]). There-



[d] Def. 6. [b] and 8. 7. [e] 26. 1. [f] Def. 6. 7.

Therefore B*bac*CA is equal to the triple of the pyramid PQRU [g], ſtanding on an equal [h] baſe (PQR). And, in the ſame manner, the priſm A*acd*DC, on the baſe ACD, is equal to the triple of the pyramid PRSU, on the equal baſe PRS; and ſo on. Therefore, alſo, ſhall the whole priſm A*d*, on the baſe ABCDE, be equal to the triple of the whole pyramid PQRSTU, on the equal baſe PQRST.

[g] Cor. to 7. 8.

[h] Hyp. and Ax. 10.

COROLLARY I.

Hence, all pyramids having the ſame baſe and altitude, are equal; being like parts of one and the ſame priſm.

COROLLARY II.

Hence, alſo, all priſms having the ſame baſe and altitude, are equal; being equimultiples of one and the ſame pyramid.

COROLLARY III.

Therefore it appears, that every priſm inclining on its baſe, as well as every upright one, is equal to a rectangular parallelepipedon of equal baſe and altitude [k]; and, conſequently, that all priſms whatever, having equal baſes, and altitudes, are equal to each other [k]: which muſt be alſo true in pyramids and cones, every ſuch ſolid being ſubtriple to a priſm, or cylinder, of the ſame baſe and altitude [l].

[k] 20. 7. and Ax. 1.

[l] 8. 8. and 6. 8.

COROLLARY IV.

Hence it alſo follows, that whatever is demonſtrated in the 21ſt, 22d, and 23d Theorems of the preceding Book, concerning the proportion of priſms, holds equally in pyramids and cones; *theſe* being like parts of *thoſe*.

COROL-

COROLLARY. V.

It follows, moreover, that all correſponding fruſtums of pyramids and cones of the ſame altitude, are alſo, in proportion, as their baſes. For, the ſections, at all equal altitudes, being in that proportion [m], the parts cut off (as well the wholes) will be in the ſame proportion [n]; and, conſequently, the remaining parts likewiſe [o].

[m] Lem. 5.
[n] Cor. 4.
[o] 3. 4.

COROLLARY VI.

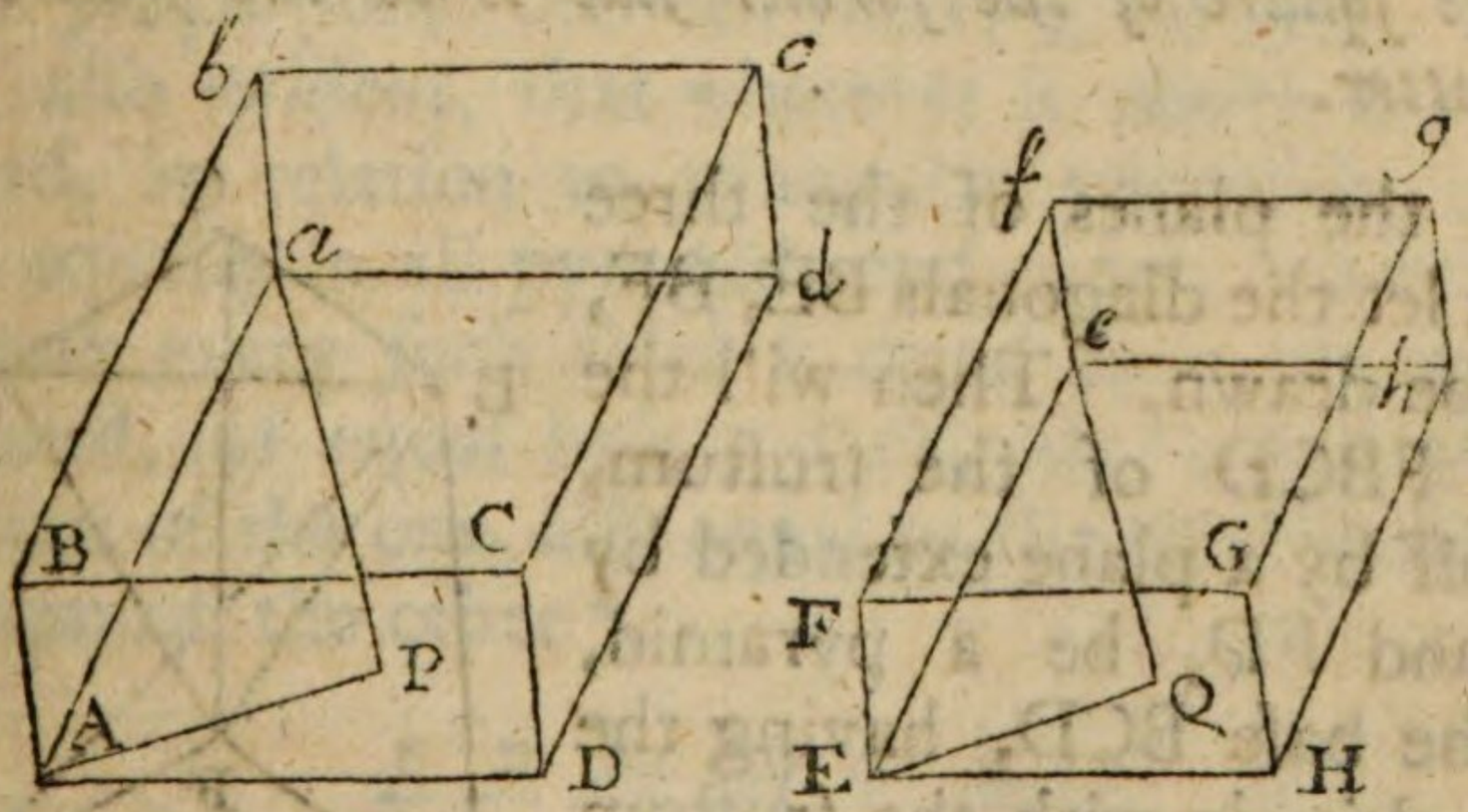
Laſtly, it will appear, that all cones, which have their altitudes and the diameters of their baſes directly proportional, are in the triplicate ratio of their altitudes [p]; being to each other in the ſame proportion with priſms of equal baſe and altitude, whereof they are like parts [q].

[p] 24. 7.
[q] Cor. 3. and Cor. to 1. 4.

THEOREM IX.

All ſimilar priſms, and pyramids, are in the triplicate ratio of their altitudes.

From the extremes of the homologous ſides A*a*, E*e*, upon the baſes ABCD, EFGH of the propoſed ſolids A*c*, E*g*, let fall the perpendiculars



*a*P, *e*Q. The angle BAD being = FEH, BA*a* = FE*e*, and DA*a* = HE*e* [p], thence is *a*P : *e*Q : : *a*A : *e*E [q] : : AB : EF [r] : : AD : EH [r]. Therefore

[p] Def. 5. 7.
[q] 14. 7.
[r] Def. 14. 4.

two

two upright prisms constituted on the bases ABCD, EFGH, of the same altitudes (*a*P, *e*Q) with the two solids A*c*, E*g*, will be similar, the one to the other [p]; and, therefore, in the triplicate ratio of the altitudes [s]. But the solids A*c*, E*g*, when taken as prisms, are respectively equal to the said upright ones; and, when taken as pyramids, are like parts of them [t]. Therefore the solids A*c*, E*g* are also in the triplicate ratio of the altitudes *a*P and *e*Q [u].

[p] Def. 5. 7.
[s] 24. 7.
[t] Cor. 3. to 8. 8.
[u] Cor. 1. 4.

COROLLARY.

Because *a*P : *e*Q : : A*a* : E*e* : : AB : EF *&c.* it follows, that all similar prisms, and pyramids, are to one another, in the triplicate ratio of the homologous sides of the like planes by which they are bounded [y].

[y] Cor. 1. to 5. 4.

THEOREM X.

The frustum (ABCDEFA) *of any pyramid having a triangular base, is equal to a whole pyramid, of the same base and altitude, together with two other pyramids that are, in proportion thereto; the one, as any side* (BD) *of the upper base* (BCD) *is to its correspondent* (AE) *of the lower base* (AFE); *and the other, as the square of the former side is to the square of the latter.*

In the planes of the three sides, let the diagonals BE, BF, FD be drawn. Then will the part FBCD of the frustum, cut off by a plane extended by FB and FD, be a pyramid, on the base BCD, having the same altitude with the frustum itself [z], both solids being contained between the same [a] parallel planes AFE, BCD.

[z] Def. 16. 7.
[a] Def. 15. 7.

More-

Moreover, the remaining part FABDE of the fruſtum, if a plane be extended by FB and FE, will be divided into the two pyramids FBDE, FABE, having the ſame ratio, one to the other, as their baſes BED, ABE [b], or as BD to AE [c]. But the latter of theſe pyramids (BAFE), taking B as the vertex thereof, has the ſame baſe and altitude with the fruſtum given: And the pyramid FBCD (firſt mentioned) is therefore, in proportion thereto, as the baſe BCD to the baſe AFE [d], that is (becauſe the baſes are ſimilar [e]), as BD^2 to [f] AE^2: *whence the propoſition is manifeſt.*

[b] 21. 7. and Cor. 4. to 8. 8.
[c] 7. 4.
[d] 21. 7. and Cor. 4. to 8. 8.
[e] Cor. to Lem. 5.
[f] 26. 4.

COROLLARY.

Since, of the three ſolids (FABE, FBDE, FBCD) into which the propoſed fruſtum is divided, the ratio of the firſt and third, is the duplicate of that of AE to BD, or of the ratio of the firſt to the ſecond [g]; it is evident, that theſe three ſolids are proportionals [h]. From whence it appears, that the fruſtum of any triangular pyramid is equal to two (whole) pyramids of the ſame altitude, on baſes equal to the two oppoſite baſes of the fruſtum, and to a third pyramid, which is a mean proportional between the two former. And it is alſo evident, that whatever is above demonſtrated, in relation to triangular pyramids, muſt hold equally in all pyramids and cones, whatever: Becauſe every ſuch ſolid is equal to a triangular pyramid, of equal baſe and altitude [i]; and every fruſtum of the one, alſo equal to the correſponding fruſtum of the other [k].

[g] Cor. to 27. 4.
[h] Def. 7. 4.
[i] Cor. 3. to 8. 8.
[k] Cor. 5. to 8. 8.

LEMMA 6.

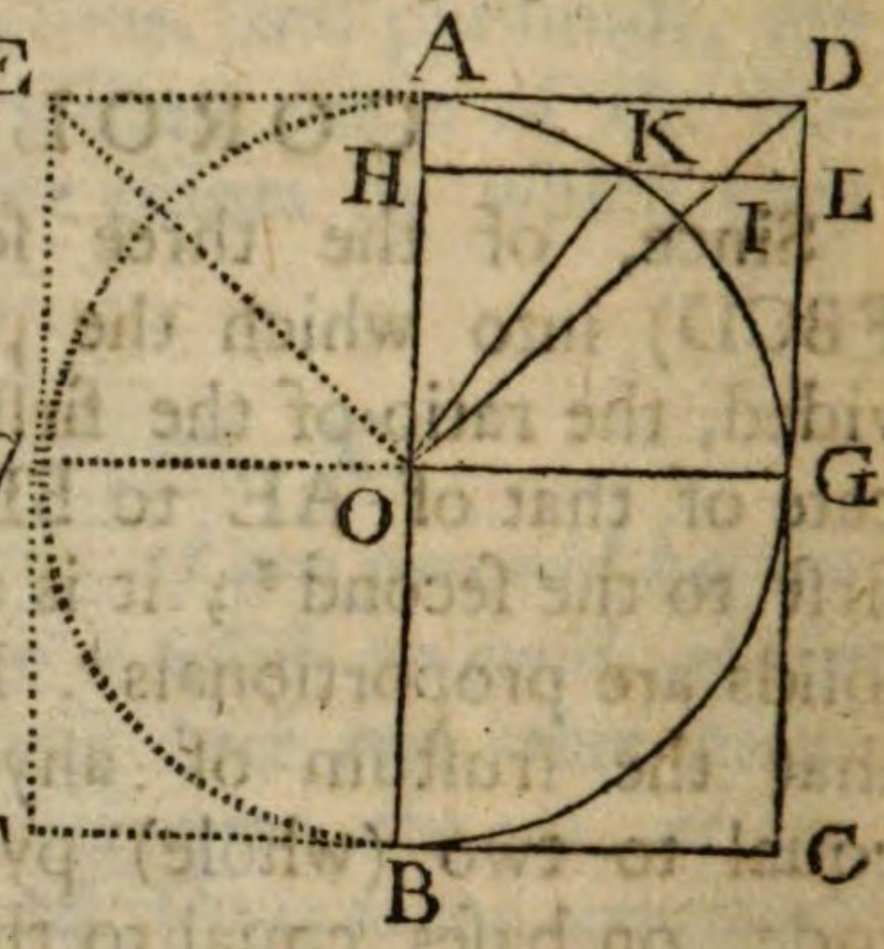
If with radii, reſpectively equal to the three ſides of any right-angled triangle, three circles be deſcribed; that whoſe radius is equal to the hypothenuſe, will be equal to both the other two, taken together.

It

It has been proved (*in Theor.* 3.) that circles are in proportion, as the ſquares of their radii; therefore the demonſtration here, is the very ſame, as in ſimilar right-lined figures (*Theor.* 29. *Book* 4.): Which (if neceſſary) you may conſult.

THEOREM XI.

Every ſphere is two-thirds of its circumſcribing cylinder (*Or, of a cylinder of equal diameter and altitude.*)



Let AB be the axis about which the ſphere and cylinder are generated, by the revolution of the ſemi-circle AGB and the rectangle AIDCB[1]; let HL be any right-line perpendicular to AB, meeting DC in L, and the periphery of the ſemi-circle in K; and from the center O, let OK and OD, interſecting HL in I, be drawn.

[1] Def. 12. and 14.

Since AO is = AD[m], and HI parallel to AD[n], therefore is HI = OH[o]: But OHI being right-angled at H, the circle whoſe radius is OH (or HI) will (*by the preceding Lem. and Ax.* 5.) be equal to the difference of the two circles whoſe radii are OK (HL) and HK: Or, in other words, the circle deſcribed by HI, or the ſection of the cone generated by the triangle AOD, in its revolution about the axis AB, will be equal to the difference of the two circles generated by HL and HK; that is, equal to the *annulas* deſcribed by KL[q], or the ſection of the ſolid which remains, when the ſphere is taken out of the cylinder. Therefore ſeeing theſe

[m] Hyp.
[n] 4. 1.
[o] 14. 4.
[p] Lem. 6.
[q] Ax. 2. 1.

theſe two ſections are, every-where, equal to each other, the ſolids themſelves will likewiſe be equal [r]; that is, the cone (EOD) will be equal to the exceſs of the cylinder (GDEg) above the inſcribed hemiſphere (GAg): whence, as the cone, or exceſs, is one third part of the cylinder [s], the hemiſphere muſt neceſſarily be equal to the two remaining thirds. And what is here proved, with reſpect to the halves of the propoſed ſolids, holds equally in the wholes. Therefore every ſphere is two-thirds of its circumſcribing cylinder.

[r] Lem. 4.

[s] 6. 8. and Cor. 3. to 8. 8.

COROLLARY I.

Hence, a cone, hemi-ſphere and cylinder, of the ſame altitude, and ſtanding upon equal baſes, are in proportion, as the numbers 1, 2 and 3, reſpectively.

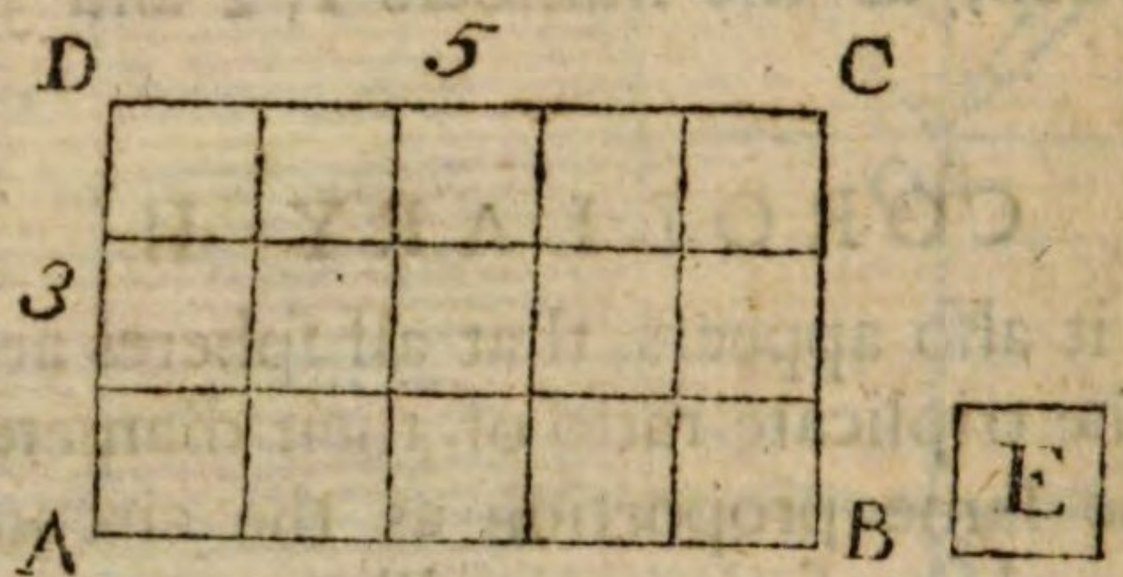
COROLLARY II.

Hence it alſo appears, that all ſpheres are to each other in the triplicate ratio of their diameters [t]; being in the ſame proportion as the circumſcribing cylinders, whereof they are like parts.

[t] Cor. to 5. 8. and 24. 7.

END *of the* ELEMENTS.

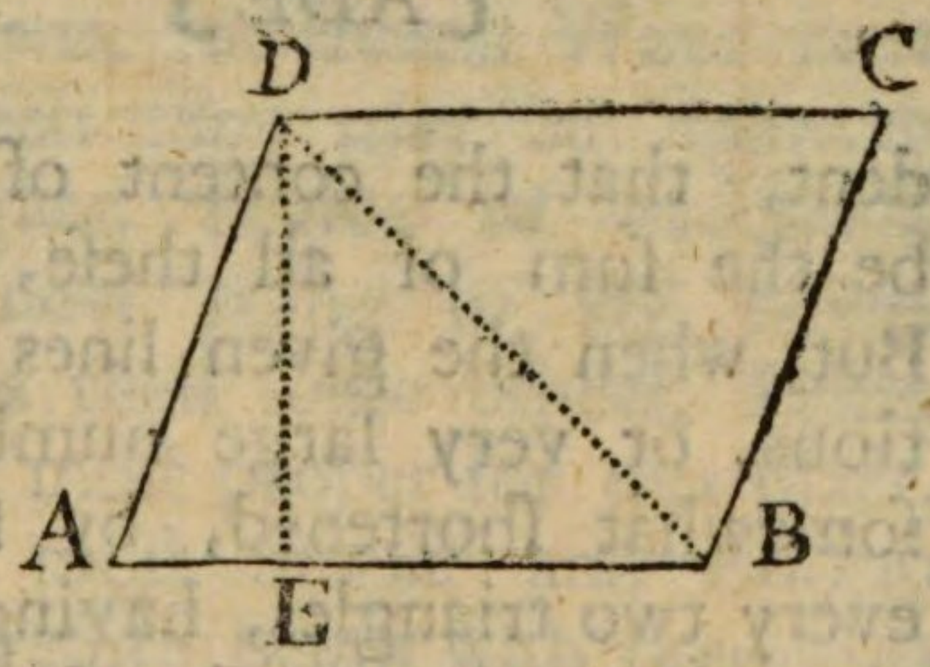
OF THE
MENSURATION
OF
Superficies and Solids.



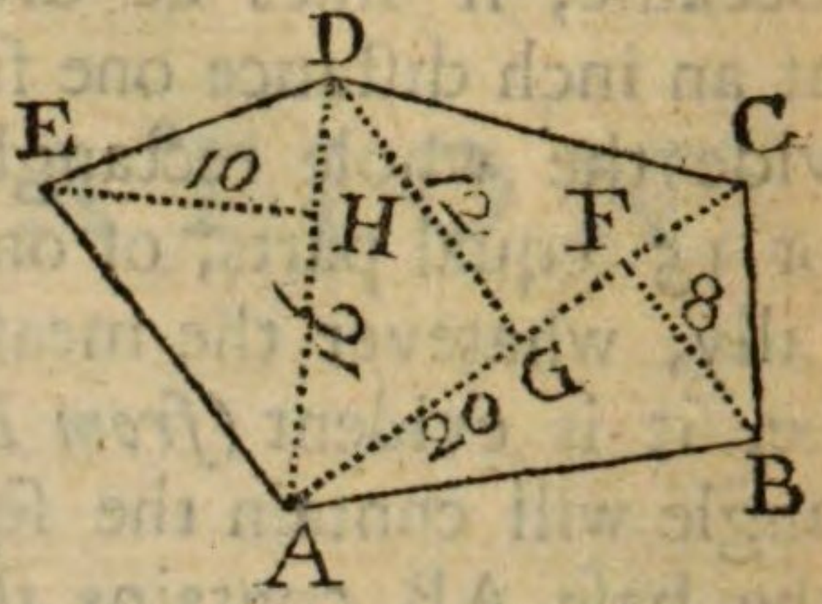
EVERY quantity is measured by some other quantity of the same kind; as a line by a line, a surface by a surface, and a solid by a solid: And the number which shews how often the lesser, called *the measuring unit*, is contained in the greater, or quantity measured, is called *the content of the quantity so measured*. Thus, if the quantity to be measured be the rectangle ABCD, and the little square E, whose side is one inch, be the measuring unit propounded; then, as often as the said little square is contained in the rectangle, so many square inches the rectangle is said to contain: So that, if the length DC be supposed 5 inches, and the breadth AD 3 inches; the content of the

rectangle

rectangle will be 3 times 5, or 15 ſquare inches: Becauſe, if lines be drawn parallel to the ſides, at an inch diſtance one from another, they will divide the whole rectangle ABCD into 3 times 5, or 15 equal parts, of one inch each. And, generally, whatever the meaſures of the two ſides may be, it is evident *(from El. 7. of 4.)* that the rectangle will contain the ſquare E, as many times as the baſe AB contains the baſe of the ſquare, repeated as often as the altitude AD contains the altitude of the ſquare. Therefore, *to find the content of any rectangle, multiply the baſe by the altitude, and the product will be the anſwer.* Thus, let the length be 18 inches, and the breadth 15; then the content will be 15 times 18, or 270 ſquare inches.



The method of finding the content of a rectangle being thus known, the content of any parallelogram ABCD, or triangle ABD, will alſo be known; the former of theſe figures being equal to a rectangle of the ſame baſe and altitude; and the latter equal to the half of ſuch a rectangle (by Cor. 2. to 2. 2.) Therefore, *multiply the baſe by the perpendicular, for the content of any parallelogram; and the baſe by half the perpendicular, for that of any triangle.* Thus, for example, let the baſe AB be 18 feet, and the perpendicular DE 12 feet; then the content of the parallelogram will be 216, and that of the triangle 108, ſquare feet.



From the manner of finding the area of a triangle, the area of any right-lined plain figure, as ABCDE, may be determined, by dividing the whole into triangles and finding the content of each triangle. Thus, let the dividing lines AC and AD, be 20 and 16 inches, and the perpendiculars BF, DG, EH, falling thereon, 8, 12, and 10, reſpectively; then, the content of the

$$\text{Triangle} \begin{Bmatrix} ABC \\ ACD \\ ADE \end{Bmatrix} \text{ being } \begin{Bmatrix} 80 \\ 120 \\ 80 \end{Bmatrix},$$ it is evi-

dent, that the content of the whole figure will be the ſum of all theſe, or 280 ſquare inches. But, when the given lines are expreſſed by fractions, or very large numbers, the work will be ſomewhat ſhortened, by finding the content of every two triangles, having the ſame baſe, at one operation; that is, by firſt adding the two perpendiculars together, and then multiplying half their ſum by the common baſe of the two triangles. Thus, in the laſt example, the half-ſum of the two perpendiculars BF and DG being 10, if this number be, therefore, multiplied by 20 the meaſure of the common baſe AC, the product, which is 200, will be the content of the trapezium ABCDA; to which 80, the content of the triangle ADE, being added; the ſum will be 280, the ſame as before. But, if the polygon propoſed be a regular one, that is, one whoſe ſides, and angles are all equal, the ſhorteſt

way

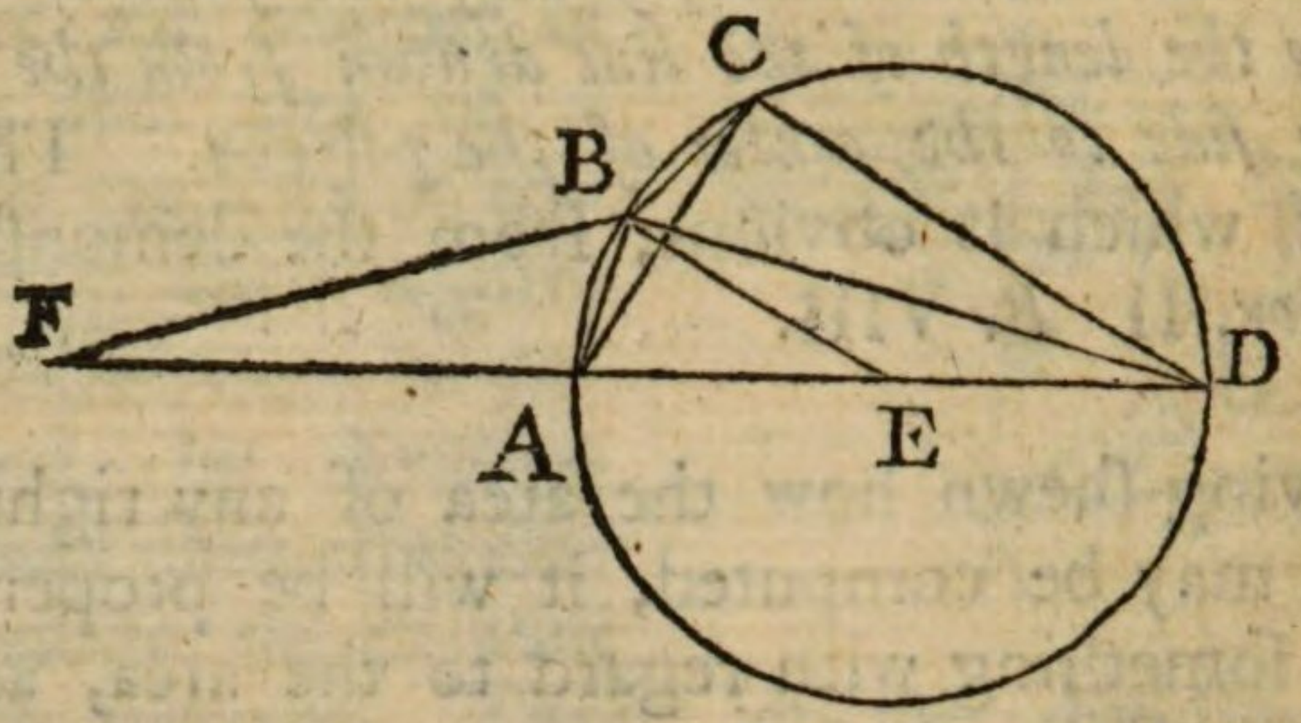
way of all, is, to *multiply half the ſum of all the ſides by the length of the line drawn from the middle of any ſide to the center of the polygon.* The reaſon of which is obvious, from the demonſtration to *Theor.* II. *B.* VIII.

Having ſhewn how the area of any right-lined figure may be computed, it will be proper here, to ſay ſomething with regard to the area, and periphery, of the circle.

It is well known, that to determine the *true* area of a circle, and to find a right-line *exactly* equal to the circumference thereof, are looked upon, by mathematicians, as abſolutely impoſſible: But, though neither the one nor the other can be *accurately* known, yet ſeveral Ways have been invented by which they may be approximated, to any aſſigned degree of exactneſs. That, which I am now going to lay down, though leſs expeditious than ſome others, ſeems, nevertheleſs, to be the moſt proper for this place, as depending on the moſt ſimple and evident principles: I ſhall therefore begin with premiſing the following

LEMMA.

If AD *be a diameter, and* AB, BC *two equal arcs of the ſame circle, and if the chords* DB, DC *be drawn; then, I ſay, that* $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$.



For, if in DA produced, there be taken AF = DC, and BF, BA, BC and the radius BE be drawn; then, the external angle FAB, of the trapezium ABCD, being equal to the internal oppoſite angle DCB *(by* 17. 3.) alſo AF = DC, and AB = CB *(by Hyp.)*; it is evident, that FB is alſo = DB, and conſequently the angle F = FDB = DBE: And ſo the iſoſceles triangles DEB, DBF being equiangular, it will be as DE ($\frac{1}{2}$AD): DB :: DB : DF (DC + AD); and conſequently $DB^2 = \frac{1}{2}AD \times DC + \frac{1}{2}AD^2$. *Q. E. D.*

COROLLARY.

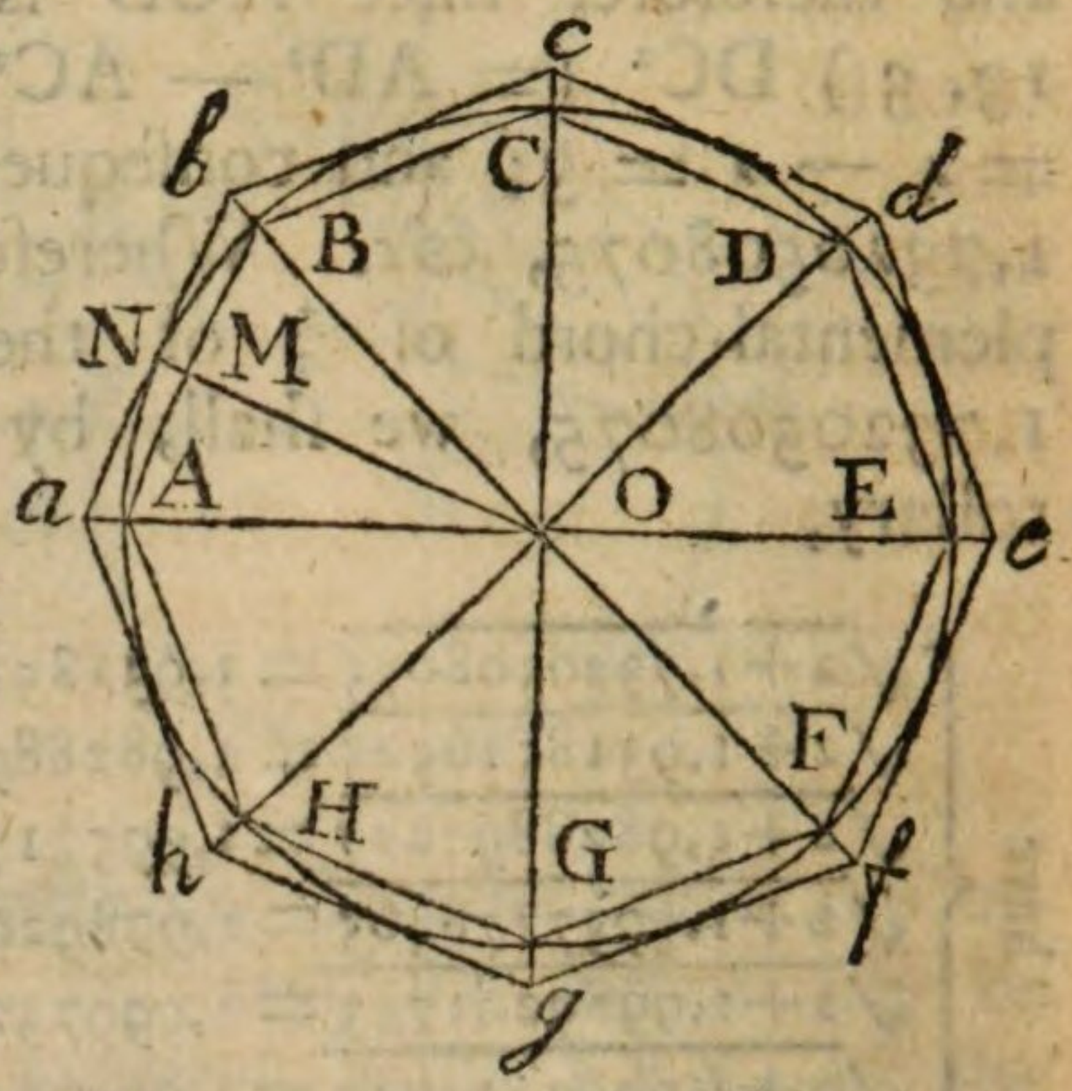
Hence, if the diameter AD be denoted by the number 2, the chord DB will be denoted by $\sqrt{DC + 2}$: whence, it appears, that, *if the meaſure of the ſupplemental-chord of any arch be increaſed by the number* 2, *the ſquare-root of the ſum will be the ſupplemental-chord of half that arch.*

Now, to apply this to the matter propoſed, that is, to the finding of the area and circumference of the circle; let the arch ABC be taken equal to $\frac{1}{3}$ of the ſemi-periphery ACD; then will the chord AC be equal to the radius AE *(by* 29. 5.); and

and therefore, since ACD is a right-angle (*by* 13. 3.) DC^2 ($= AD^2 - AC^2$, *by* 8. 2.) will be $= 4 - 1 = 3$; and consequently $DC = \sqrt{3} =$ 1,7320508075, *&c.* Wherefore, seeing the supplemental-chord of $\frac{1}{3}$ of the semi-periphery is 1,7320508075, we shall, by the preceding Corollary,

have { $\sqrt{2+1,7320508075} = 1,9318516525$; $\sqrt{2+1,9318516525} = 1,9828897227$; $\sqrt{2+1,9828897227} = 1,9957178465$; $\sqrt{2+1,9957178465} = 1,9989291743$; $\sqrt{2+1,9989291743} = 1,9997322757$; $\sqrt{2+1,9997322757} = 1,9999330678$; $\sqrt{2+1,9999330678} = \sqrt{3},9999330678$ } for the supplemental chord of { $\frac{1}{6}$; $\frac{1}{12}$; $\frac{1}{24}$; $\frac{1}{48}$; $\frac{1}{96}$; $\frac{1}{192}$; $\frac{1}{384}$ } of the semi-periphery.

Now, therefore, since it is found that 3,9999330678 is the square of the supplemental-chord of $\frac{1}{384}$ of the semi-periphery, let this number be subtracted from 4 the square of the diameter, and the remainder 0,00000669322 will be the square of the chord of the same arch; therefore the chord itself being $= \sqrt{0,00000669322} = 0,00081812 1$, let this number be multiplied by 768, or twice 384, and the product 6,28317 will be the perimeter of a regular polygon of 768 sides, inscribed in the circle; which, as the sides of the polygon very nearly coincide with the circumference of the circle, must also express the length of the circumference itself, very nearly.



But, in order to ſhew how near this is to the truth, let AB repreſent one ſide of a regular polygon of 768 ſides, inſcribed in the circle (whoſe length, we have found above, to be 0,00818121) and let *ab* be a ſide of another ſimilar polygon, deſcribed about the circle; and from the center O let ON be drawn, biſecting AB and *ab* in M and N: Then, ſince AM is $= \frac{1}{2}AB = 0{,}0040906$, and $AO = 1$, it is plain that OM^2 ($AO^2 - AM^2$) will be $= 0{,}99998327$, and conſequently $OM = 0{,}99999163$; whence, becauſe of the ſimilar triangles AOB, *a*O*b*, &c. we have 0,99999163 (OM) : 1 (ON) : : AB : *ab* : : 6,28317 (the perimeter of the inſcribed polygon) : 6,28322 the perimeter of the circumſcribed polygon. But the circumference of the circle being greater than the perimeter of the inſcribed polygon, and leſs than that of the circumſcribed one, it muſt, conſequently, be greater than 6,28317, and leſs than 6,28322; and muſt, therefore, be equal to 6,2832, very near; ſince this number exceeds the perimeter of the inſcribed polygon by no more than 0,00003, and is leſs than the perimeter of the circumſcribed one by 0 00002, only.

From the periphery thus found, the area of the circle will alſo be known; being equal to the product of half the periphery into the radius (*by* 2. 8.) that is, $= 3{,}1416 \times 1 = 3{,}1416$.

There-

Therefore, ſince it is proved (*in Theor.* 3. *and* 4. *of* 8.) that the peripheries of circles are in proportion as their diameters, and the circles themſelves as the ſquares of thoſe diameters; it follows, that, as 2 is to 6,2832, or as 1 to 3,1416:: the diameter of any circle to its periphery; and as 4 to 3,1416, or as 1 to 0,7854:: the ſquare of the diameter to the area.

But, if you had rather have the proportions in whole numbers, and the caſe propoſed does not require any great degree of accuracy; then, inſtead of the foregoing, thoſe of *Archimedes* may be uſed, *viz.* 7 : 22 :: diam. : circumf. and 14 : 11 :: ſquare diam. : area. Which proportions differ but little from thoſe above, as will appear from the following example: wherein the diameter of a circle being given 28, its circumference and area are required. Here, according to the firſt proportions, I multiply 28 by 3,1416 for the circumference, and the ſquare of 28 (or 784) by 0,7854 for the area; and there reſults 87,964 and 615,75, reſpectively. But, according to the proportions of *Archimedes*, the circumference will be found equal to 88, and the area 616; which differ very little from the former.

By knowing the proportion between the diameter of a circle and the circumference, and between the ſquare of the diameter and the area, the convex ſuperficies of ſolid bodies may be determined. Thus,

The convex ſuperficies of a cylinder is found, *by firſt finding the circumference of the baſe, and then multiplying by the altitude of the ſolid.* Therefore, if to that product, the area of the two circular ends be added, the ſum will be the whole ſuperficies of the cylinder.

To find the convex ſuperficies of a cone, *multiply half the length of the ſlant ſide thereof by the circumference of the baſe.*

The convex ſuperficies of any fruſtum of this ſolid is found, by *multiplying the ſum of the peripheries of the two ends into half the length of the ſlant ſide of the fruſtum.*

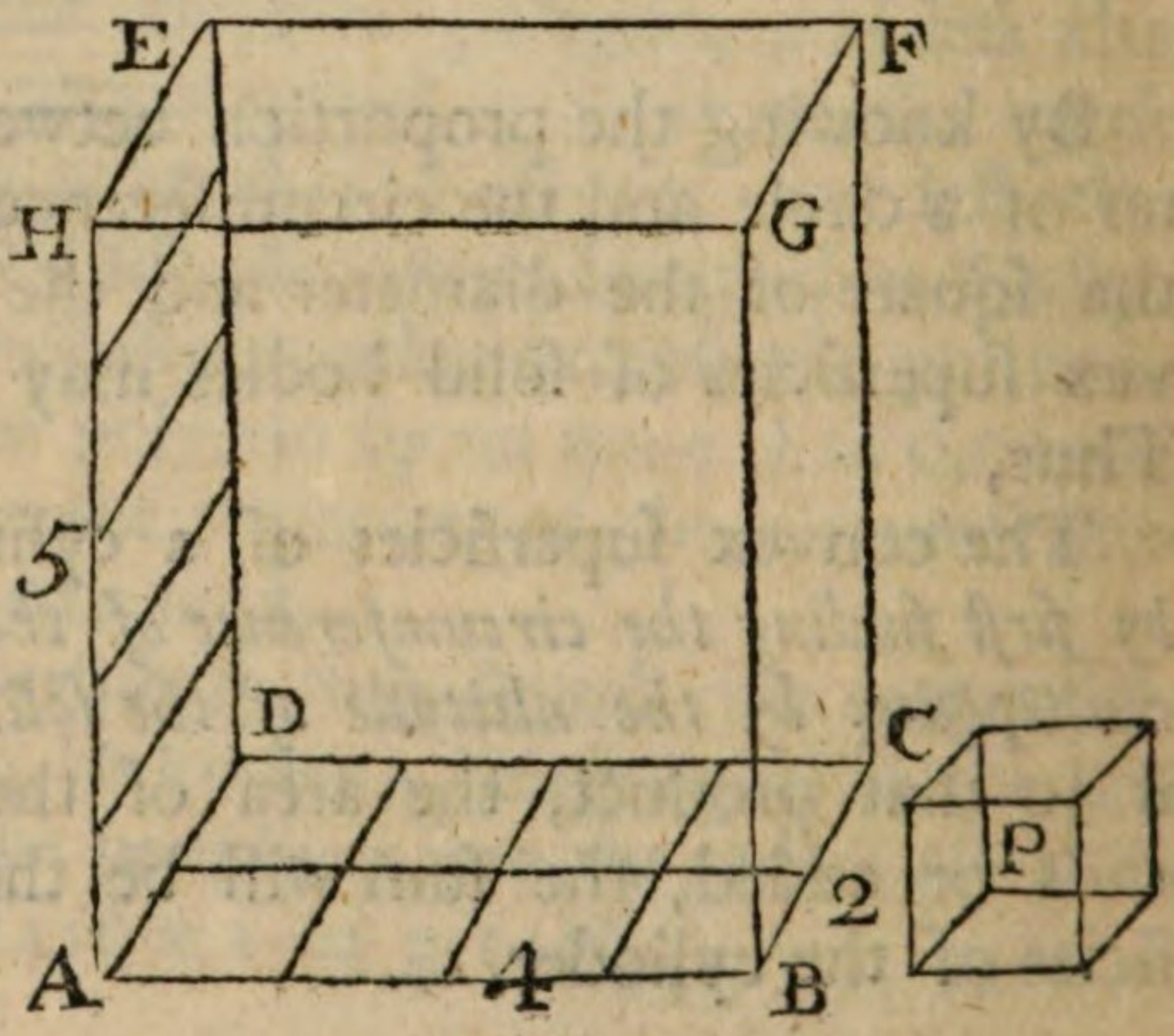
To find the ſuperficies of a ſphere, *multiply the periphery of the greateſt, or generating, circle by its diameter:* Or, *multiply the ſquare of the diameter by* 3,1416.

The convex ſuperficies of any ſegment of a ſphere is found, *by multiplying the periphery of the greateſt circle of the ſphere into the altitude of the ſegment.*

The demonſtration of theſe laſt rules, for finding the curve ſurfaces of ſolid bodies (which is not given in the *Elements*, for reaſons mentioned hereafter) is inſerted at the end of this ſection.

Of the Menſuration of Solids.

As every ſuperficies is meaſured by a ſquare, whoſe ſide is unity (as one inch, one foot, one yard, *&c.*) ſo every ſolid is meaſured by a cube whereof the ſide is alſo an unit. Thus,



let

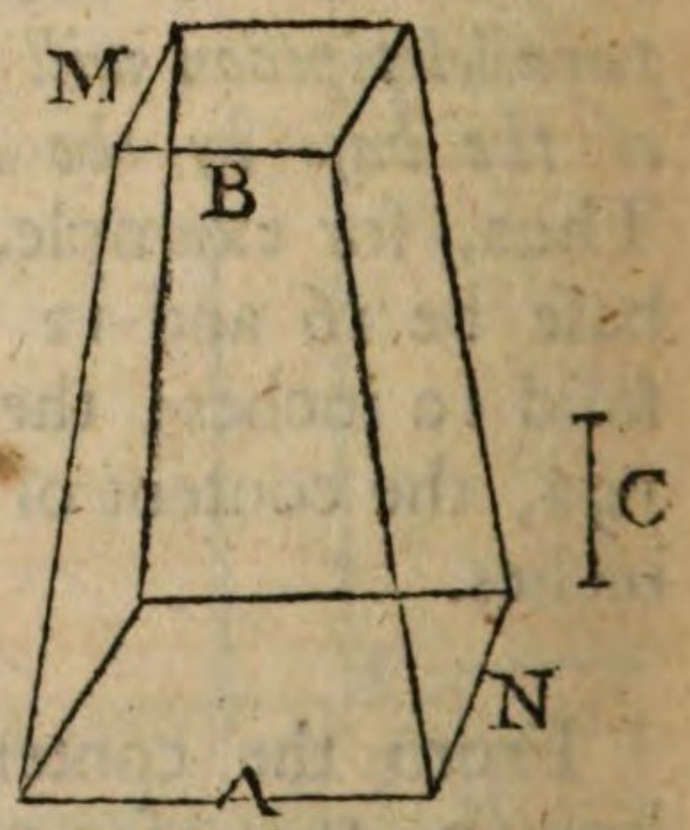
let the ſolid to be meaſured, be the rectangular parallelepipedon AF, and let the cube P, whoſe ſide is one inch, be the *meaſuring unit*; alſo let the length AB, of the baſe AC, be 4 inches, the breadth BC 2 inches, and the altitude AH of the ſolid 5 inches: Then, becauſe the area of the baſe ABCD is 2 times 4 (or 8) ſquare inches, it is eaſy to conceive, that, if the ſolid were to be only one inch high (inſtead of 5), the content thereof would be juſt the ſame number (8) of cubical inches; becauſe then, upon the eight equal ſquares into which the whole baſe ABCD is diviſible, a cube of one inch might be erected, ſo as to compoſe a parallelepipedon on that baſe, of one inch high. Therefore, ſeeing that the content of the ſolid, at one inch high, is 8 cubical inches, the whole content at 5 inches high, muſt conſequently be 5 times 8, or 40 cubical inches (ſince the whole ſolid AF may be conſidered, as compoſed of 5 ſuch heights of cubes, one ranged above another). And, generally, whatever the dimenſions may be, it is manifeſt (*from* 21 and 22. of 7.) that the parallelepipedon will contain the cube P, as many times as the baſe ABCD contains the baſe of the cube, repeated as often as the altitude AH contains the altitude of the cube. Therefore *the content of any parallelepipedon will be found, by multiplying the area of the baſe by the altitude of the parallelepipedon.* Thus, for example, if the two dimenſions of the baſe be 16 and 12 inches, and the height of the ſolid 10 inches; then, the area of the baſe being 192, the content of the ſolid will be 1920 cubical inches.

From the content of a parallelepipedon, thus known, *that* of a priſm, or a cylinder, will likewiſe be known; every ſuch ſolid being (*by* 20. 7. *or* 5. 8.) equal to a parallelepipedon of equal baſe, and

and altitude. Therefore, *multiply the area of the baſe* (found by the rules for ſuperficies) *into the height of the priſm, or cylinder, and the product will be the content.*

Hence the content of any pyramid, or cone, is alſo obtained; being (*by Cor.* 3. *to* 8. 8.) equal to $\frac{1}{3}$ part of a priſm, or cylinder, of the ſame baſe and altitude. Therefore, *multiply the area of the baſe by* $\frac{1}{3}$ *of the altitude, and the product will be the anſwer.*

Every ſphere being (*by* 11. 8.) equal to $\frac{2}{3}$ parts of a cylinder of the ſame diameter and altitude; *the content of any ſphere will, therefore, be found, by multiplying the area of its greateſt, or generating, circle into* $\frac{2}{3}$ *of its diameter:* Or (becauſe the area of ſuch circle is to the ſquare of the diameter, in proportion as 0,7854 to 1), *let the cube of the diameter be multiplied by the fraction* ,5236 (= $\frac{2}{3}$ of 0,7854), *and the product will be the content.* Thus, if the meaſure of the diameter be 20, the cube thereof will be 8000; which, multiplied by ,5236, will give 4188,8 for the meaſure of the ſphere's ſolidity.

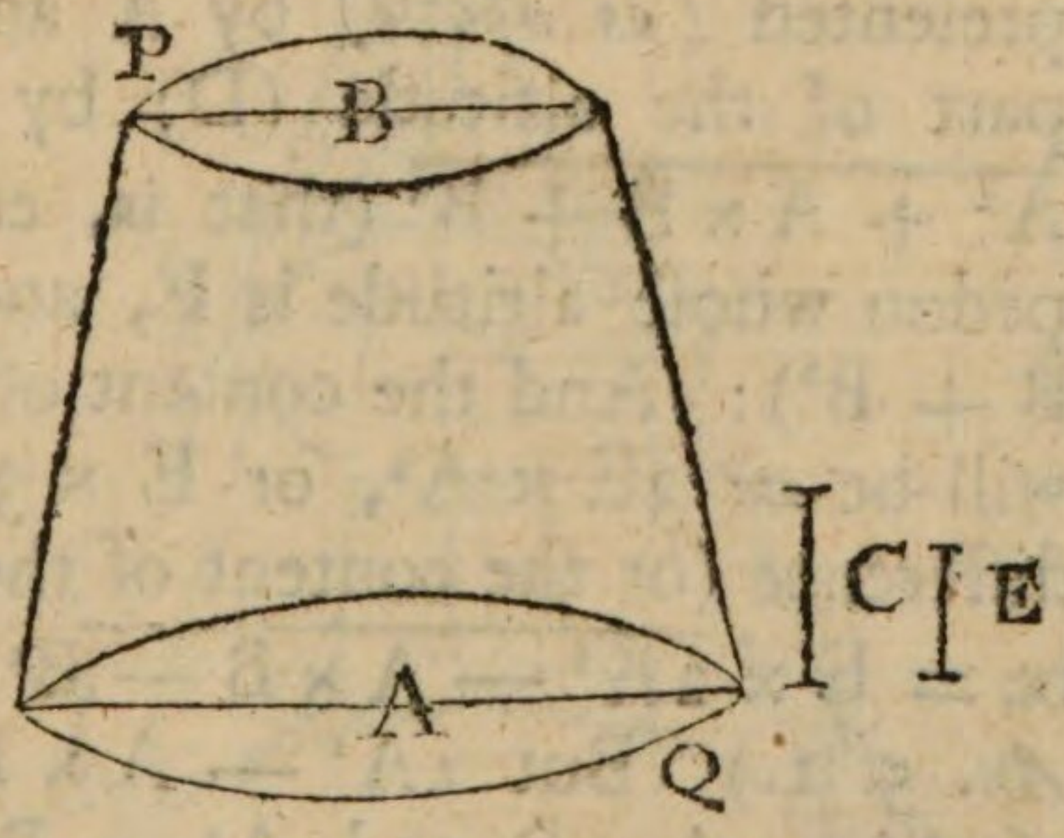


The manner of finding the content of any fruſtums of the ſolids above determined, is collected from *Theor.* 10. *and* 11. *B.* VIII. Let the fruſtum (MN), firſt propoſed, be that of a pyramid; then, having found the content of a whole pyramid, of the ſame given baſe and altitude; ſay, as any ſide A of the lower end or baſe, is to its correſpondent B of the upper, ſo is the ſaid con-

tent

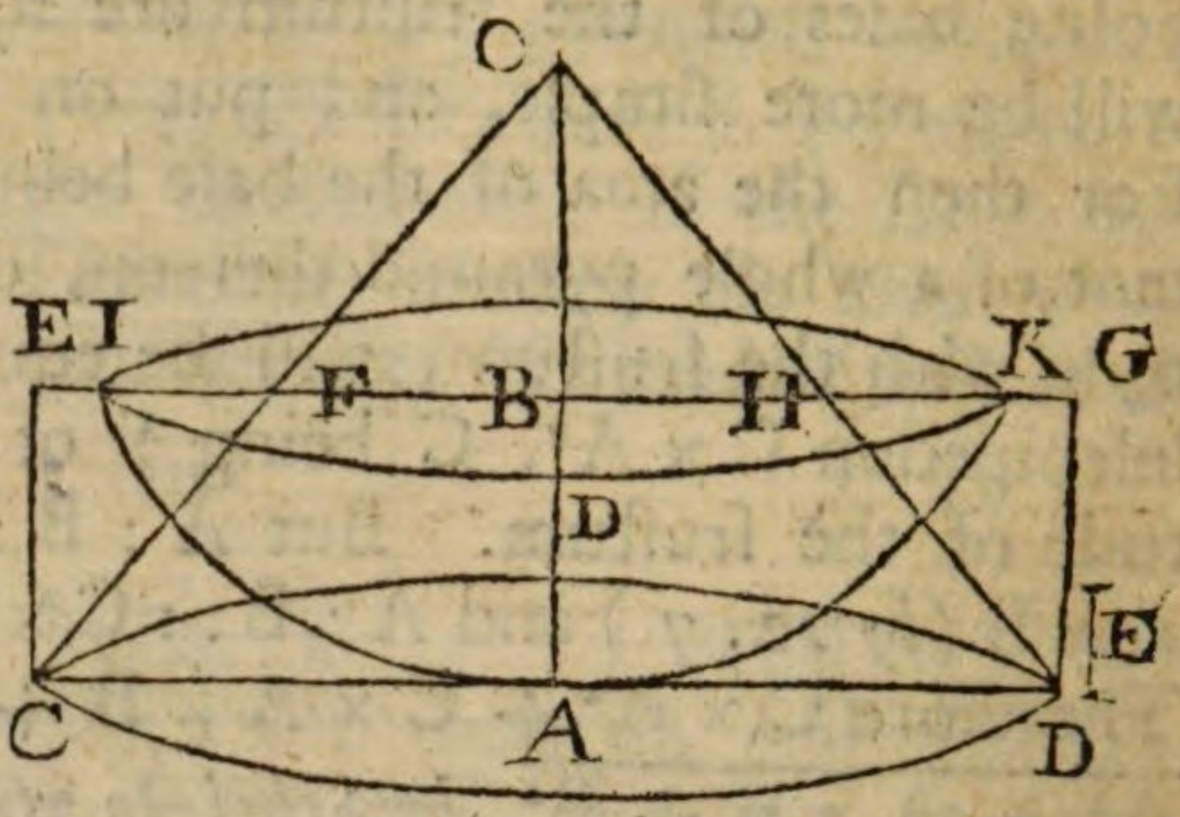
content to a fourth-proportional; and, as A is (again) to B, ſo is the quantity laſt found to another proportional: which two proportionals, added to the content firſt determined, will give the true content of the fruſtum. But when the oppoſite baſes of the fruſtum are ſquares, the rule will be more ſimple, and put on a better form: For then the area of the baſe being A^2, the content of a whole pyramid thereon, of the ſame altitude with the fruſtum, will be equal to the parallelepipedon $C \times A^2$, C being $\frac{1}{3}$ of the given altitude of the fruſtum. But $A : B :: C \times A^2 : C \times A \times B$ (*by* 22. 7.) and $A : B :: C \times A \times B : C \times B^2$. Therefore $C \times A^2 + C \times A \times B + C \times B^2$ ($= C \times \overline{A^2 + A \times B + B^2}$, *by Schol. to* 20. 7.) is the true content, in this caſe.

Hence, *to find the content of the fruſtum of any ſquare-pyramid, add the product of the two ſides of the lower and upper ends to the ſum of their ſquares, and then multiply the aggregate by* $\frac{1}{3}$ *of the pyramid's height.*



From the content here found, *that* of any conical fruſtum (PQ) is readily obtained; being in proportion to the content ($C \times \overline{A^2 + A \times B + B^2}$) of the fruſtum of a ſquare pyramid circumſcribing it, as the baſe of the former is to the baſe of the latter (*by Cor.* 5. *to* 8. 8.), or as the fraction ,7854 is to *unity*: And ſo, will be equal to the ,7854 part of $C \times \overline{A^2 + A \times B + B^2} = E \times \overline{A^2 + AB + B^2}$; by taking $E = ,7854 \times C =$ the ,618 part of the whole given altitude. Therefore

fore, *to find the content of any fruſtum of a cone, add the product of the diameters of the two ends to the ſum of their ſquares; then multiply the aggregate by the fruſtum's height, and the product, again, by the fraction* ,2618.



Hence, and from *Theor.* 11. *B.* VIII. a Rule for finding the content of any ſegment IAK of a ſphere, may alſo be deduced: For, it appears, *from thence*, that the ſegment propoſed, IAK, is equal to the difference between a conical fruſtum FCDH and a cylinder ECDG of the ſame altitude, ſtanding upon a baſe, whoſe radius CA is equal to that (AO) of the ſphere itſelf. But the content of the fruſtum FCDH, if the two diameters CD, FH be repreſented *(as above)* by A and B, and the ,2618 part of the altitude (D) by E, will be $= E \times \overline{A^2 + A \times B + B^2}$ (that is, equal to a parallelepipedon whoſe altitude is E, and baſe $= \overline{A^2 + A \times B + B^2}$): And the content of the cylinder ECDG will be $= 3E \times A^2$, or $E \times 3A^2$. Therefore the difference (or the content of the ſegment IAK) will be $= E \times \overline{2A^2 - A \times B - B^2}$ (*Schol. to* 20. 7. *and Ax.* 5. 1.) But $2A^2 - A \times B - B^2$ is compoſed of $A^2 - A \times B$ and $A^2 - B^2$; whereof the former part $A^2 - A \times B$ is $= \overline{A - B} \times A$ (*by* 5. 2.) $= 2D \times A$ (becauſe $A - B$ (or $CD - FH$) $= FF + HG = 2AB$ or $2D$); and the latter $\overline{A^2 - B^2} = \overline{A - B} \times \overline{A + B}$ (*by* 7. 2.) $2D \times \overline{2A - 2D}$: Whence the ſum of both will conſequently be $= 2D \times \overline{3A - 2D}$; and the content of the ſegment itſelf

itſelf $= E \times 2D \times \overline{3A - 2D} = ,5236 \times D^2 \times \overline{3A - 2D}$ (becauſe $2E = ,5236\,D$).

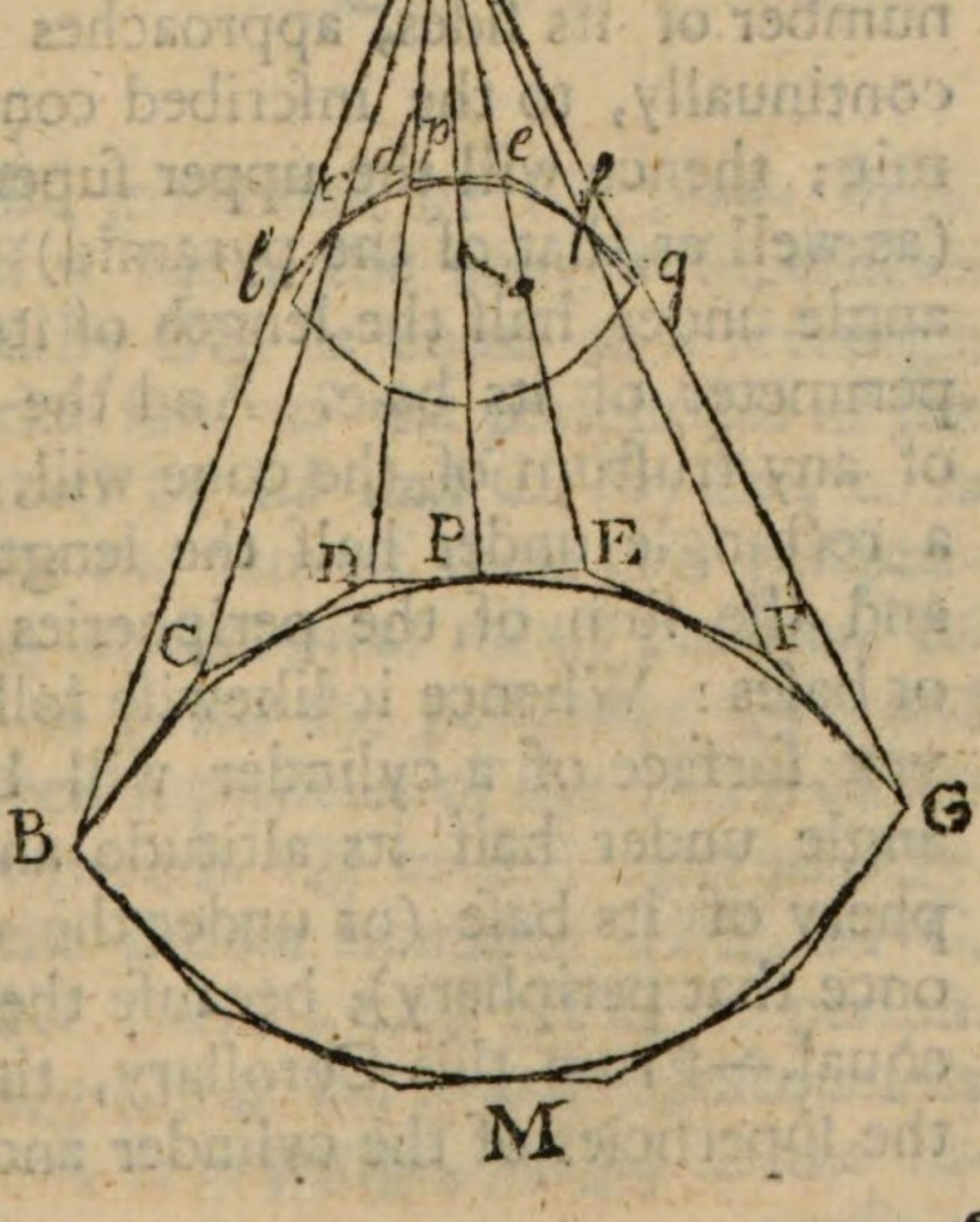
Therefore, *to find the content of any ſegment of a ſphere, multiply the ſquare of the ſegment's height by the exceſs of thrice the ſphere's diameter above the double of that height; and then multiply by the fraction* ,5236.

The demonſtration of the rules for determining the ſuperficial content of the cylinder, cone and ſphere, and of their ſeveral ſegments, or fruſtums, is collected from the two *Lemmas* here ſubjoined.

LEMMA 1.

The upper ſuperficies, or the area of all the ſides of a regular pyramid, in which a cone may be inſcribed, is equal to a rectangle under the perimeter of the baſe and half the length of the cones's ſlant ſide.

For, let BCDE, &c. be the baſe of the pyramid, and BPGM that of the inſcribed cone; and from the vertex A to the point P where any ſide DE of the polyg. touches the circle, let AP be drawn. Then, ſince the triangle ADE is $= \frac{1}{2}AP \times DE = \frac{1}{2}AB \times DE$; and as the like holds good with regard to every other ſide



of

of the pyramid, it is evident that the ſum of all the ſides, or the whole ſuperficies of the pyramid (excluſive of the baſe) will be equal to $\frac{1}{2}$AB × DE + EF + &c; that is, equal to a rectangle under $\frac{1}{2}$AB and the whole perimeter of the baſe.

COROLLARY I.

Hence it will alſo appear, that all the ſides of any fruſtum Bg of the pyramid, will be equal to a rectangle under half the length of each ſide and the ſum of the perimeters of the two ends: For, the area of the ſide DE*ed* being = $\frac{1}{2}$P*p* × DE + *de*, or $\frac{1}{2}$B*b* × DE + *de* (*by* 4. 2.), the area of all the ſides will, therefore, be = $\frac{1}{2}$B*b* × DE + *de* + EF + *ef* + &c.

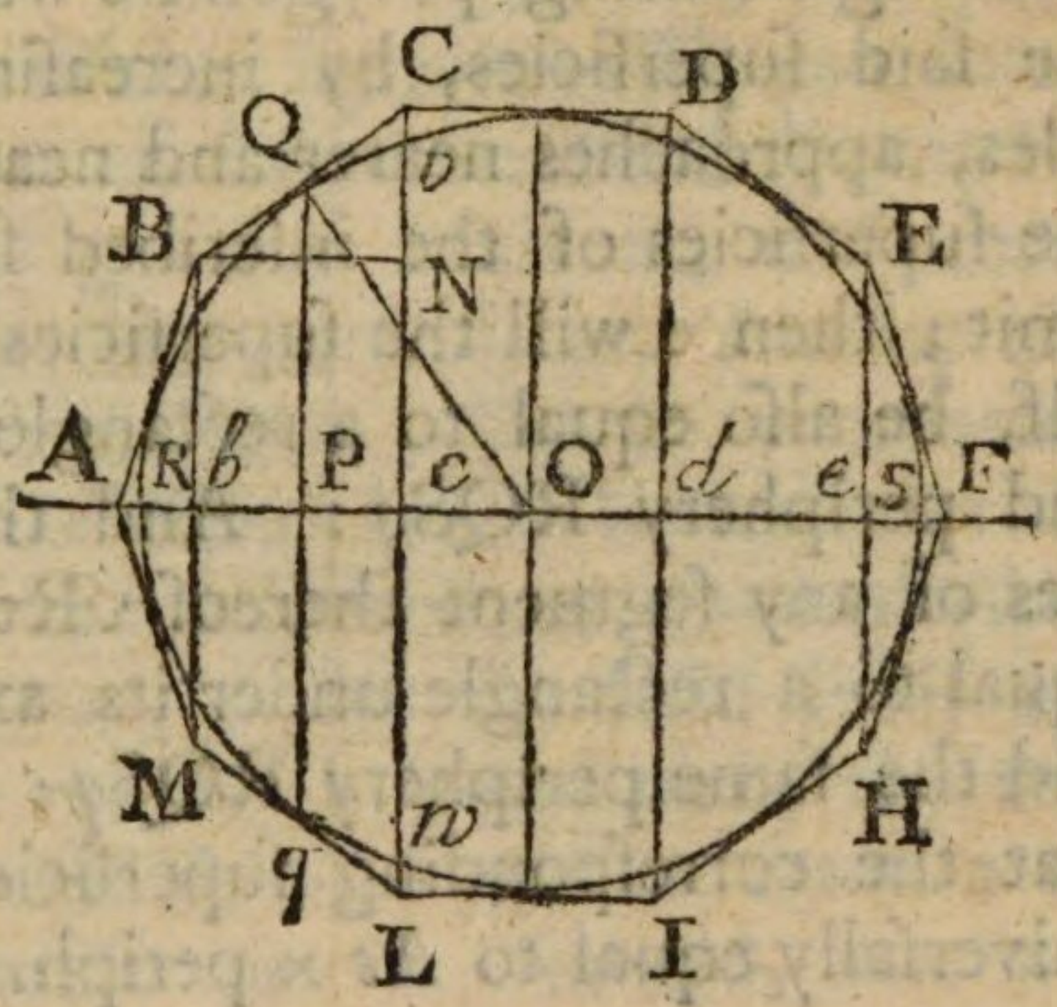
COROLLARY II.

Therefore, ſeeing that the foregoing concluſions hold univerſally, whatever the number of the ſides may be; and as the pyramid, by increaſing the number of its ſides, approaches nearer and nearer, continually, to the inſcribed cone, which is its limit; thence will the upper ſuperficies of the cone (as well as that of the pyramid) be equal to a rectangle under half the length of its ſlant ſide and the perimeter of its baſe. And the convex ſuperficies of any fruſtum of the cone will, *alſo*, be equal to a rectangle under half the length of its ſlant ſide and the ſum of the peripheries of its two ends, or baſes: Whence it likewiſe follows, that the convex ſurface of a cylinder will be equal to a rectangle under half its altitude and twice the periphery of its baſe (or under the whole altitude and once that periphery); becauſe then the two ends are equal.—From this Corollary, the rules for finding the ſuperficies of the cylinder and cone, are given.

LEM-

LEMMA 2.

If a regular polygon ABCDE, &c. *of an even number of ſides, together with its inſcribed circle* RQSq, *be ſuppoſed to revolve about the (produced) diameter* RS, *as an axis; the ſuperficies of the ſolid generated by the polygon, will be equal to a rectangle under its axis* AF *and a right line equal to the circumference* RQSq *of the inſcribed circle.*



From the center O, to the point of contact Q, of any ſide BC, let the radius OQ be drawn; alſo draw B*b*M, Q*Pq*, C*c*L, &c. perpendicular to AF, and BN perpendicular to CL.

Becauſe the ſolid generated by the plane B*bc*C is the fruſtum of a cone, the convex ſuperficies thereof, generated by BC, is equal to a rectangle under $\frac{1}{2}$BC and the ſum of the peripheries of the two circles deſcribed by B*b* and C*c* (*by Cor.* 2. *to the precedent*): But the ſum of theſe two peripheries, as QP is an arithmetical mean between B*b* and C*c*, is equal to twice the periphery Q*q*; and therefore the convex ſuperficies of the ſaid fruſtum equal to $\frac{1}{2}$BC × 2 periph. Q*q* = BC × periph. Q*q*. But, becauſe of the ſimilar triangles OPQ, BNC, we have BC : BN (*bc*) : : OQ : PQ : : periph. RQS*q* : periph. Q*q* (*by* 4. 8.); and conſequently BC × periph. Q*q* = *bc* × periph. RQS*q* = the ſuperficies

perficies generated by BC. By the very ſame argument, the ſuperficies generated by any other ſide CD is $= cd \times$ periph. RQSq : Whence it is manifeſt, that the ſuperficies of the whole ſolid is $=$ $Ab + bc + cd + \&c. \times$ periph. RQS$q = AF \times$ periph. RQSq.

COROLLARY I.

Since the ſuperficies of the ſolid is, univerſally, equal to AF $\times$ periph. RQSq, let the number of ſides of the generating polygon be what it will ; and as the ſaid ſuperficies, by increaſing the number of ſides, approaches nearer and nearer, continually, to the ſuperficies of the inſcribed ſphere, which is its limit ; thence will the ſuperficies of the ſphere, itſelf, be alſo equal to a rectangle under its axis RS and periphery RQSq : And the convex ſuperficies of any ſegment thereof vRw, will likewiſe be equal to a rectangle under its axis (or height) Rc and the ſame periphery RQSq ; ſince it is proved, that the correſponding ſuperficies of CBAML, is univerſally equal to A$c \times$ periph. RQSq.

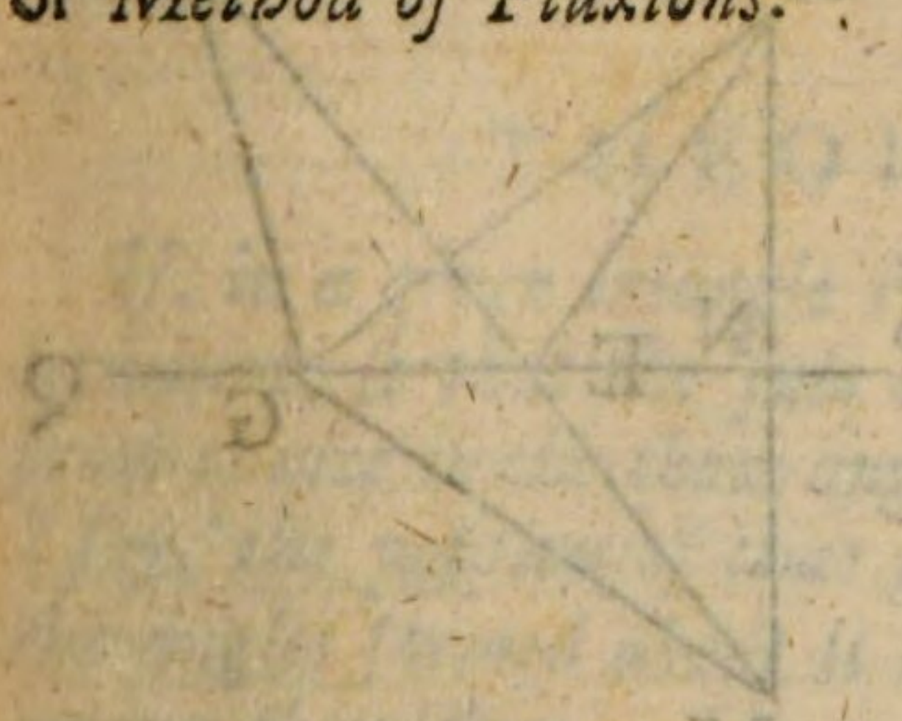
COROLLARY II.

Hence it alſo appears, that the ſuperficies of every ſphere is equal to four times its generating circle : Becauſe (*by* 2. 8.) the circle RQS$q = \frac{1}{2}$RS $\times \frac{1}{2}$ periph. RQS$q = \frac{1}{4}$RS $\times$ periph. RQSq.

In deriving theſe concluſions, as well as thoſe depending upon the preceding Lemma, the *Reader* muſt have obſerved, that ſomething is aſſumed, which is not demonſtrated in any part of theſe *Elements*. But this will not, I imagine, be conſidered as a fault, by Thoſe who know, that it is impoſſible to prove in a manner *perfectly* regular and geometrical, that a *curve ſurface*, of any kind, is equal

to

to a *plane-one* of an assigned magnitude. Plane surfaces are compared with one another, in virtue of the 10*th Axiom*; in which, whatever relates to the equality of plane figures, has its original. But no principles have been yet admitted into the *common*, or *lower Geometry*, whereby a curve-surface can be compared with a plane one; nor even by which the proportion of any one curve-line to a right-line can be known: Nor can it be demonstrated by all the Geometry in *Euclid's Elements*, that the periphery of a circle is less than the perimeter of its circumscribing square.—We can determine the proportion of solids bounded by curve-surfaces, by describing other solids in, and about them, so as to differ less from them, than by any assigned part however small. But in comparing of the surfaces, this method fails; because, let the number of sides of the inscribed, or circumscribed solid be ever so great, or let the solid itself approach ever so near to the proposed one; the two surfaces, after all, *will have no part in common* on which a demonstration can be formed, but will still be distinct things. Before such a comparison can possibly be made, in a regular and scientific manner, *new principles* must be laid down: But *these* belong to, and are best supplied in the *Modern Geometry*, or *Method of Fluxions*.



O O F

OF THE

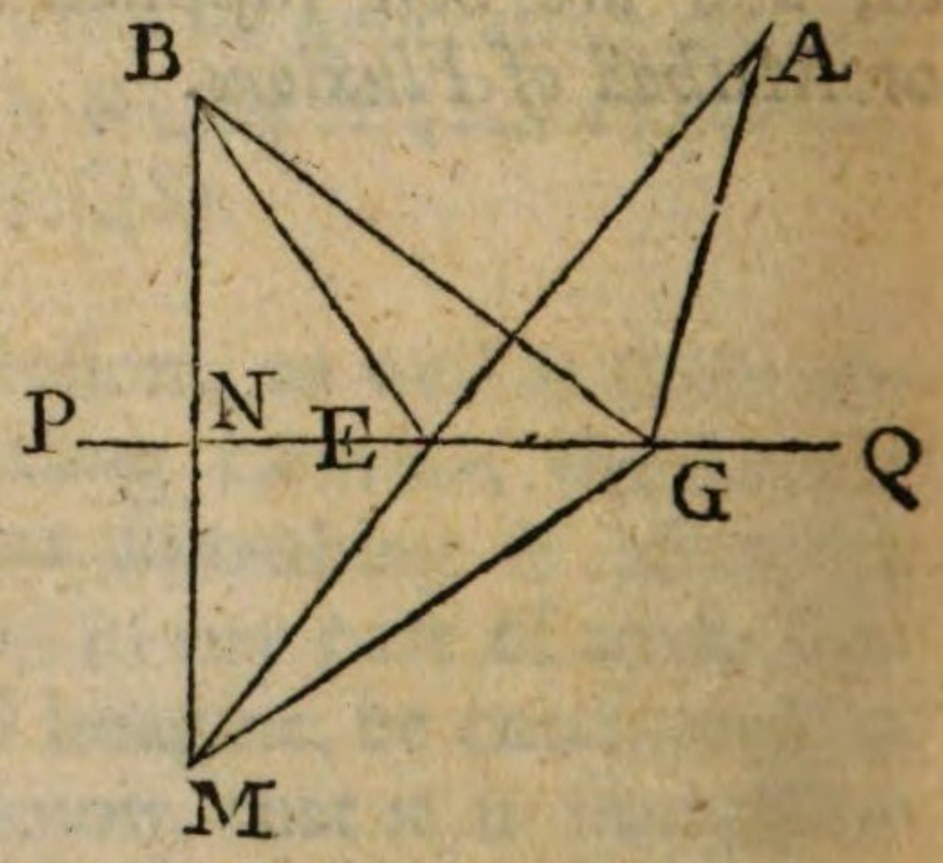
MAXIMA and MINIMA

OF

Geometrical Quantities.

THEOREM I.

If from two given points A, B, *on the ſame ſide of an indefinite line* PQ *(in the ſame plane with them) two lines* AE, BE *be drawn to meet on, and make equal angles* AEQ, BEP *with the ſaid line* PQ; *the lines ſo drawn, taken together, ſhall be leſs than any other two* AG, BG, *drawn from the ſame points to meet on the ſame line* PQ.



For, let BNM be perpendicular to PNQ, and let AE be produced to meet it in M, alſo let MG be drawn.

Then the triangles MNE, BNE, having the angle MEN (= AEQ[a]) = BEN[b], MNE = BNE[c], and NE common to both; have alſo MN = BN, and ME = BE

[a] 3. 1.
[b] Hyp.
[c] Conſtr.

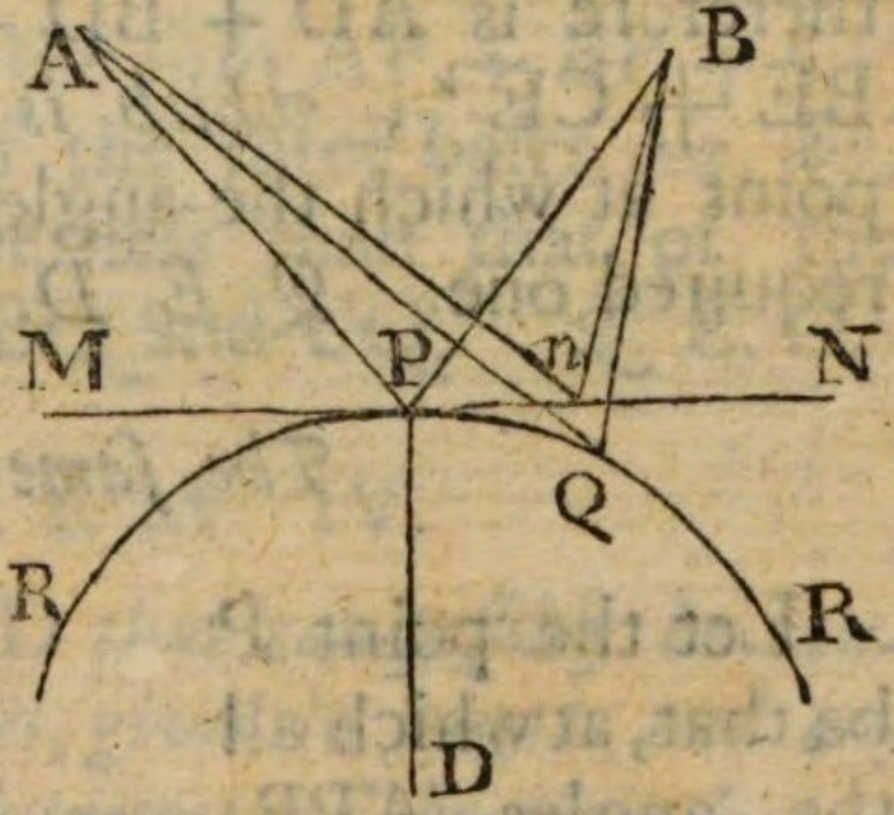
BE [d]; whence also MG = BG [e]: But AM (AE + BE) is less than AG + MG [f], or, than its equal AG + BG. *Q. E. D.*

[d] 15. 1.
[e] Ax 10. 1.
[f] 19. 1.

THEOREM II.

Of all right lines AP, BP; AQ, BQ, *that can be drawn from two given points* A, B, *to meet, two by two, on the convexity of a given circle* RPQR; *those two* AP, BP *taken together, shall be the least, which make equal angles with the tangent* MPN (*or with the radius* DP) *at the point of concourse* P.

For, if to any point *n* in the part of the tangent intercepted by AQ and BQ, there be drawn A*n* and B*n*; then will AP + BP be less than A*n* + B*n* [g], and A*n* + B*n* less than AQ + BQ [h]: Consequently AP + BP is less than AQ + BQ. *Q. E. D.*



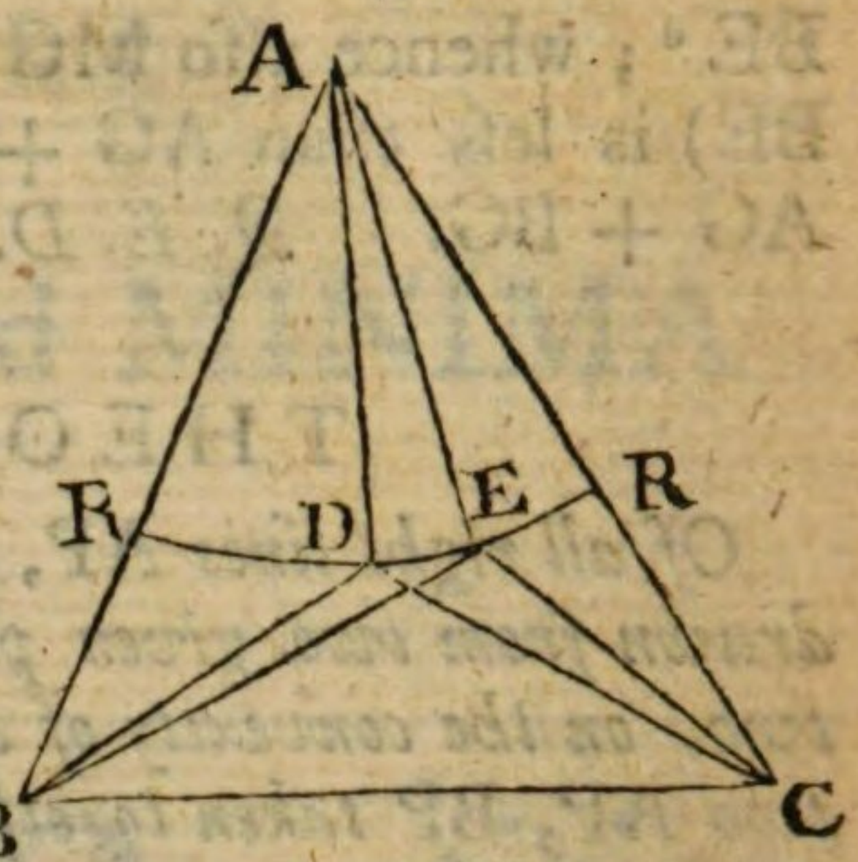
[g] Theor. 1.
[h] 23. 1.

This demonstration holds equally true, when the curve RPR is supposed of any other kind; provided all tangents to it, fall intirely without the curve.

THEOREM III.

If, in a given triangle ABC, *a point is to be determined, so that the sum of all the three lines drawn from thence to the three angles, shall be the least possible; I say, the position of that point must be such, that all the angles formed about it by those lines, shall be equal among themselves.*

If you deny it, then let ſome point E, at which the angles BEA, CEA are unequal, be the required one.

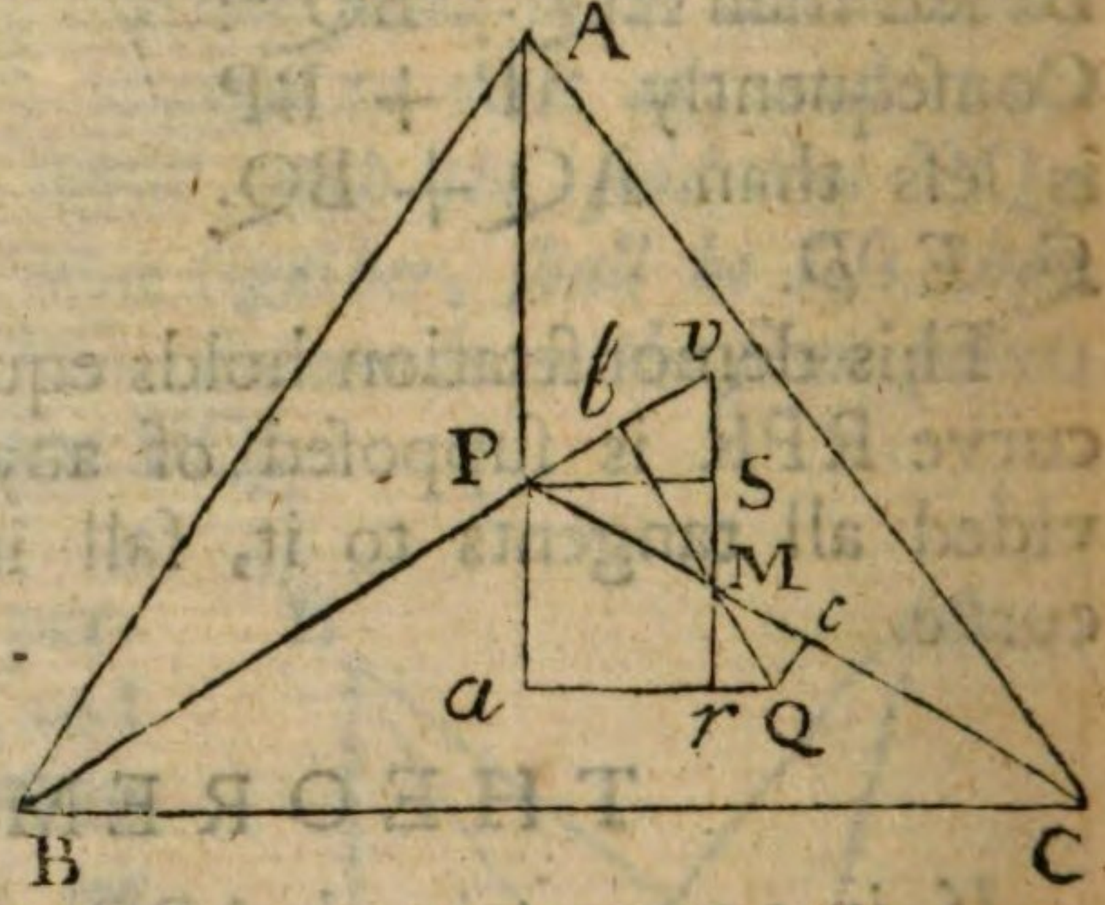


Upon the center A, thro' E, let the circumference of a circle RER be deſcribed; and let D be that point in it, where the angles ADB and ADC are equal.

[i] Theor. 2. Becauſe BD + CD is leſs than BE + CE [i], therefore is AD + BD + CD alſo leſs than AE + [k] Ax. 6. 1. BE + CE [k]; *which is repugnant.* Therefore no point at which the angles are unequal, can be the required one. *Q. E. D.*

The ſame otherwiſe.

Let the point P be that, at which all the angles APB, APC, BPC are equal *; and from any other point Q, upon the lines forming them, let fall the three perpendiculars Q*a*, Q*b*, Q*c*. I ſay, firſt, that the ſum of the three diſtances A*a*, B*b*, C*c*, intercepted by thoſe perpendiculars, and the three given points A, B, C, will be equal to the ſum of the three



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* *The determination of the poſition of a point, at which, line drawn from three given points, ſhall form any given angles, is given among the Geometrical Conſtructions, in the next ſection.*

firſt

firſt diſtances AP + BP + CP. For, if thro' the interſection M of P*c* and *b*Q, *rv* be drawn parallel to A*a*, meeting B*b* (produced) in *v*, and PS, parallel to *a*Q, in S; it is evident (becauſe the angle $v = {}^{l}BP\alpha = vPM = \frac{2}{3}$ of a right angle [m]) that the triangle P*v*M is equilateral; and that the right-angled triangles QM*c*, QM*r*, having QM*c* (= PM*b* = *v*M*b*) = QM*r*, have alſo *c*M = *r*M; to which let MP = M*v* be added; ſo ſhall *c*P = *rv* = *a*P + *v*S = *a*P + P*b*. And, if to the firſt and laſt of theſe, AP + BP + C*c* be (again) added; then will AP + BP + CP = A*a* + B*b* + C*c*, *as was aſſerted*: Whence the Theorem itſelf is exceedingly obvious: For ſeeing that the ſum AP + BP + CP is but equal to the ſum of the three baſes A*a*, B*b*, C*c*, it muſt neceſſarily be leſs than that of the three hypothenuſes AQ, BQ and CQ. *Q. E. D.*

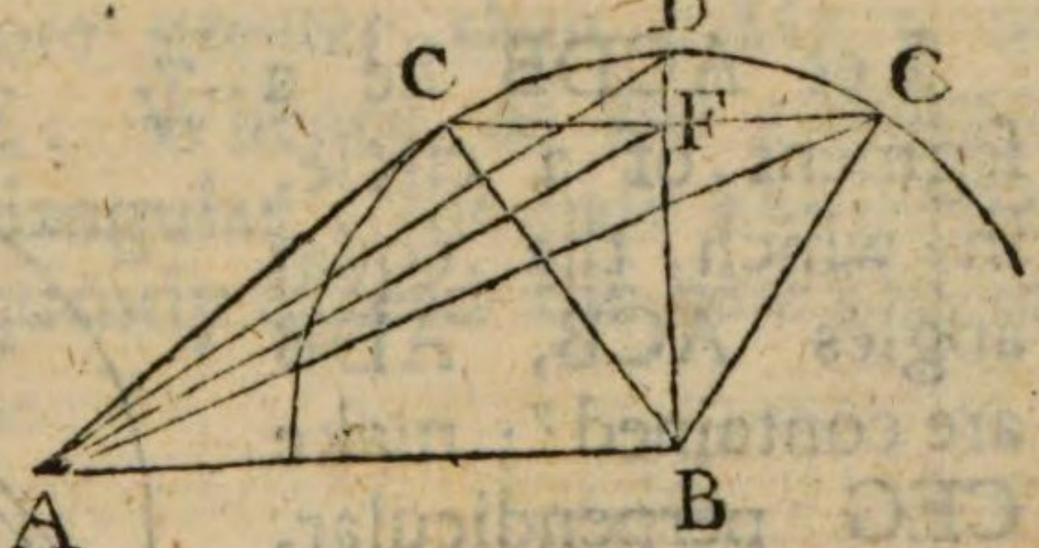
[l] Cor. to 8. 1.

[m] Hyp.

THEOREM IV.

The greateſt triangle ABD *that can poſſibly be contained under two right-lines, given in length, and any other right line joining their extremes, will be when the two given lines* AB, BD *make right-angles with each other.*

For, let BC be equal to BD, and the angle ABC either greater, or leſs than the right-angle ABD; let alſo CF be drawn parallel to AB, meeting BD (produced, if neceſſary) in F, and let A, F, and A, C be joined.



Then the angle BFC being a right-one [n], it is evident that BC (BD) is greater than BF [o]; and therefore the triangle ABD, being greater than ABF [p], is alſo greater than its equal [q] ABC. *Q. E. D.*

[n] 5. 1.

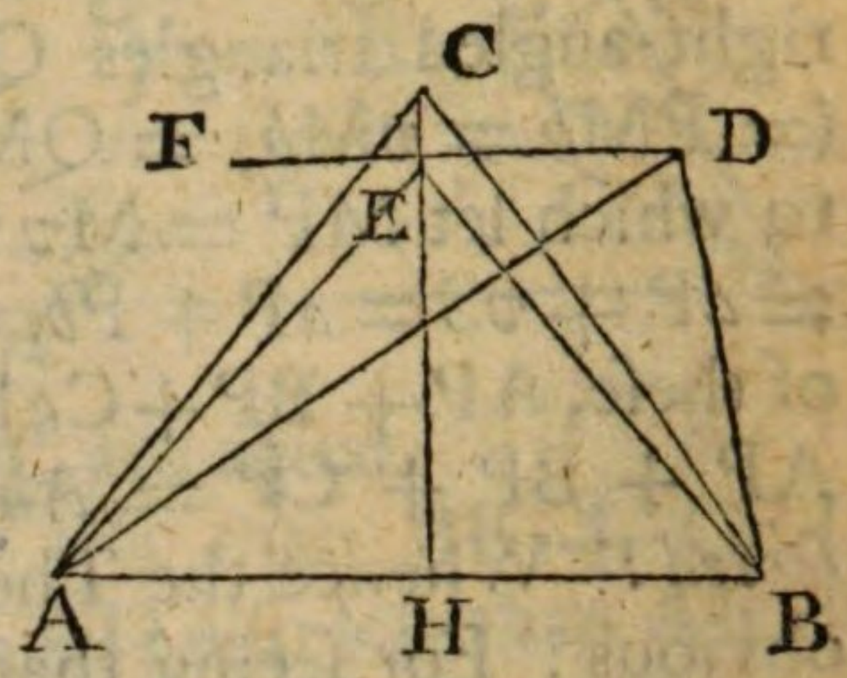
[o] 20 1.

[p] Ax. 2.

[q] Cor. to 2. 2.

THEOREM V.

Of all triangles ABC, ABD, *having the ſame baſe* AB, *and the ſum of their other ſides the ſame, the iſoſceles one* ACB, *is the greateſt.*



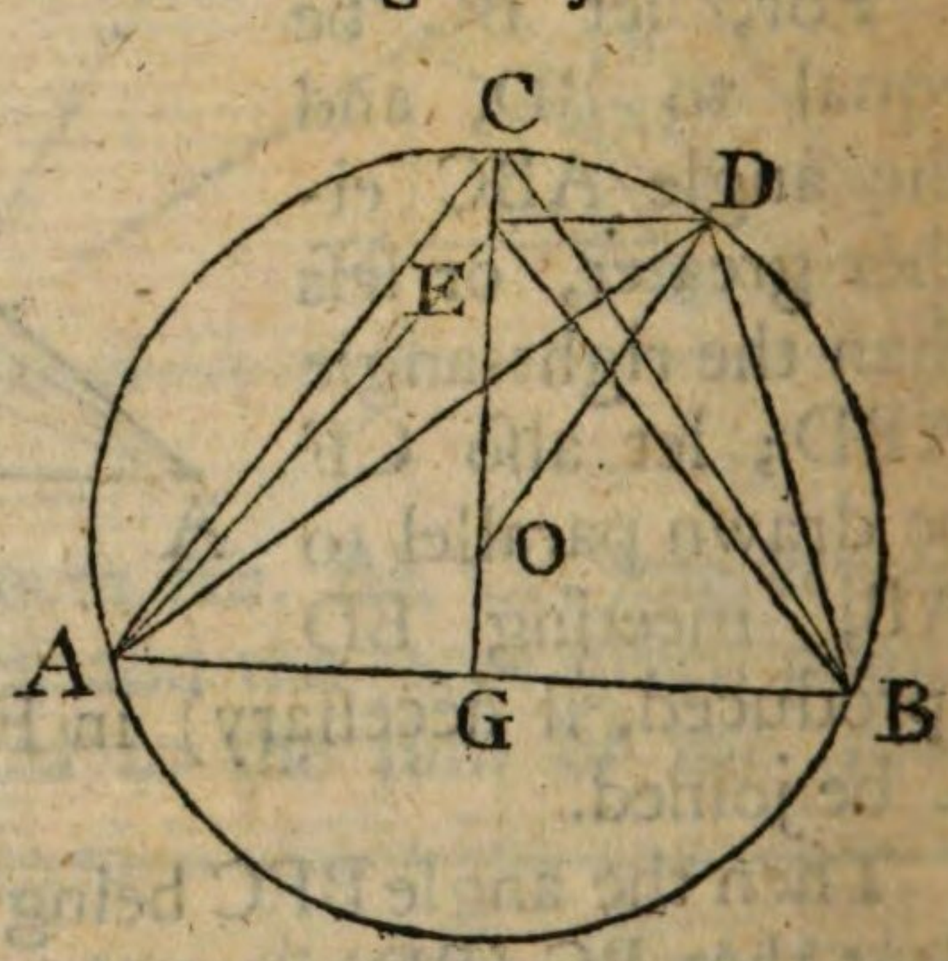
Let CH be perpendicular to AB, and DEF parallel to AB, interſecting HC (produced if need be) in E; likewiſe let AE and BE be drawn.

It is manifeſt that the angles AEF, BED are equal [r]; therefore AE + BE is leſs than AD + BD [s], or than its equal AC + BC [t]; and ſo the triangle AEB, falling within the triangle ACB [u], muſt be leſs than ACB [w]; and therefore ADB (= AEB [x]) muſt alſo be leſs than ACB. *Q. E. D.*

[r] 16. and 7. 1.
[s] Theor. 1.
[t] Hyp.
[u] 23. 1.
[w] Ax. 2. 1.
[x] Cor. 1. to 2. 2.

THEOREM VI.

Of all triangles ABC, ABD *ſtanding upon the ſame baſe* AB, *and having equal vertical angles* ACB, ADB, *the iſoſceles one* ACB *is the greateſt.*



Let ACDB be a ſegment of a circle, in which the equal angles ACB, ADB are contained [y]; make CEG perpendicular, and DE parallel, to AB; from the center O draw OD, and let A, E and B, E be joined. It is evident that CG, not only biſects AB [z], but alſo paſſes through the center O [a]. Therefore, OD (OC) being

[y] 22. 5. and 11. 1
[z] 16. 1.
[a] Cor. to 2. 3.

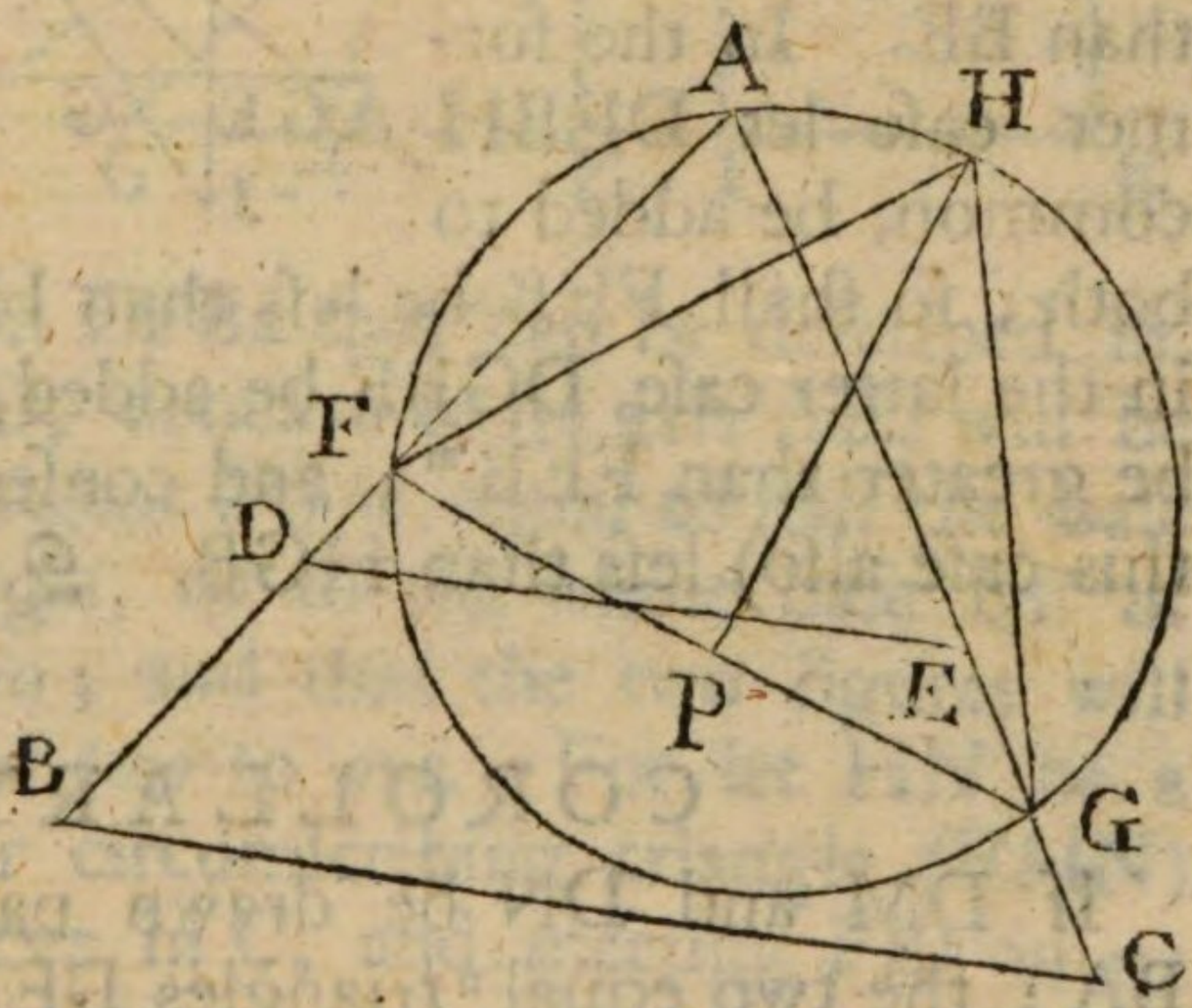
ing greater than OE [a], the triangle ACB will alſo be greater than AEB, or than its equal ADB [b]. *Q. E. D.*

[a] 20. 1.

[b] Cor. 2. 2. and Ax. 2.

THEOREM VII.

Of all right-lines DE, FG *that can be drawn to cut off equal areas* ADE, AFG *from a given triangle* ABC, *that* DE *is the leaſt, which makes the triangle* ADE *cut off an iſoſceles one.*



Let AFG be the circumference of a circle paſſing through the three points A, F, G; alſo let PH be perpendicular to FG, at the middle point P, meeting the circumference in H, and let FH and GH be drawn. The triangle FHG, being iſoſceles [c], is therefore greater than FAG [d], or than its equal [e] ADE: Whence, as the triangles FHG, ADE are equiangular [f], the baſe FG of the greater, muſt conſequently exceed the baſe DE of the leſſer. *Q. E. D.*

[c] Ax. 10.

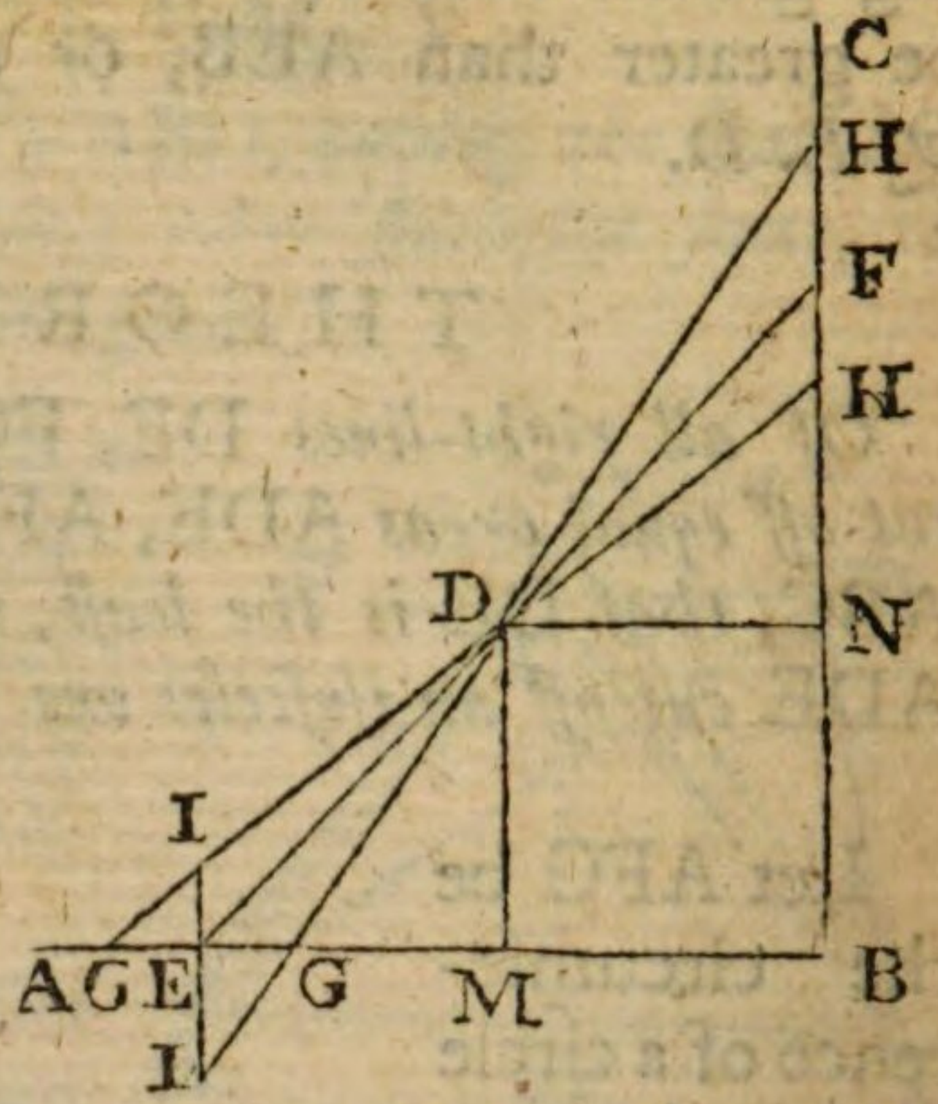
[d] Theor. 6.

[e] Hyp.

[f] Hyp. and Cor. to 10. 1.

THEOREM VIII.

Of all right-lines EF, GH, GH *that can be drawn thro' a given point* D, *between two right-lines* BA, BC *given in poſition; that* EF *which is biſected by the given point* D, *forms with them the leaſt triangle* (EBF).



For, if EI, parallel to BC, be drawn, meeting GH in I; the [h] equiangular triangles DFH and DEI, having DF = DE[i], will be equal[k]; and DFH will therefore be leſs, or greater than DEG[l], according as BG is greater or leſs than BE. In the former caſe let DEBH common, be added to both; ſo ſhall FEB be leſs than HGB[k]. And if, in the latter caſe, DGBF be added, then will HGB be greater than FEB[m]; and conſequently FEB (in this caſe alſo) leſs than HGB. *Q. E. D.*

[h] 3. and 7. 1.
[i] Hyp.
[k] 15. 1.
[l] Ax. 2.
[m] Ax. 6.

COROLLARY.

If DM and DN be drawn parallel to BC and BA; the two equal [o] triangles DEM, DFN, taken together (ſince EM = DN [o] = MB[p]) will be equal to the parallelogram DMBN[q]; and conſequently the parallelogram DMBN = ½BEF ⊐ ½BGH. Whence it is manifeſt, that a parallelogram is always leſs than half a triangle in which it is inſcribed, except when the baſe of the one is half the baſe of the other; in which caſe the parallelogram is juſt half the triangle.

[o] 15. 1.
[p] 24. 1.
[q] Cor. to 2. 2.

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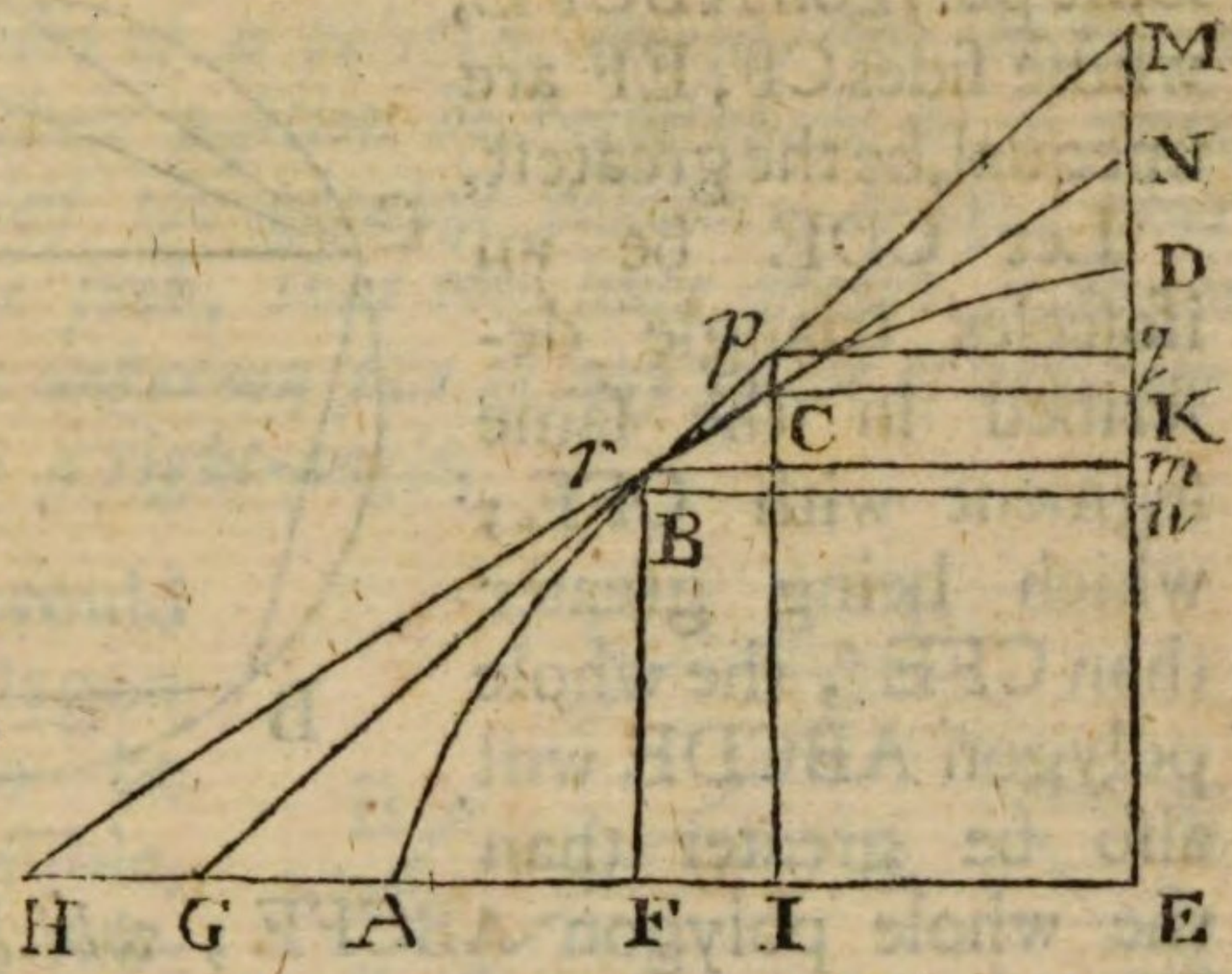


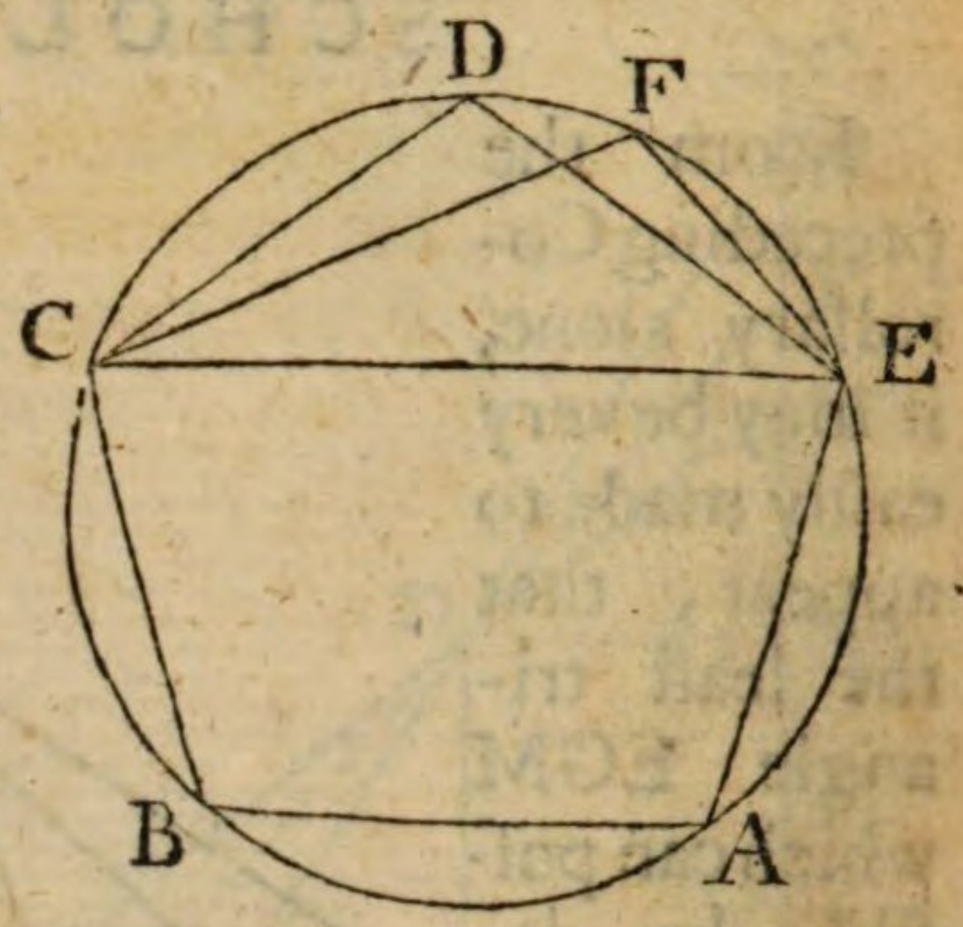
From the preceding Corollary alone, it may be very eaſily made to appear, that the leaſt triangle EGM which can poſſibly be deſcribed about, and the greateſt parallelogram EFB*n* that can be deſcribed in, any curve ABCD, concave to its axis AE, will be when the ſub-tangent FG is equal to half the baſe EG of the triangle, or to the whole baſe EF of the parallelogram; and that the two figures will be in the ratio of *two* to *one*. For let HN be a ſide of any other circumſcribing triangle (EHN) touching the curve in C, and meeting FB*r* in *r*: Then, the curve being concave to its axis, the point *r* will fall above B; whence, if *rm* be drawn parallel to B*n*, then will EGM = 2BE ⌐ 2*r*E ⌐ EHN. Again, if IC, parallel to EM, be produced to meet GM in *p*, and CK and *pq* be drawn parallel to AE; then, alſo, will BE = ¼EGM ⌐ *p*E ⌐ CE, *as was to be ſhewn.*

THEOREM IX.

Of all right-lined figures, contained under the ſame number of ſides, and inſcribed in the ſame circle, that is the greateſt whoſe ſides are all equal.

For,

For, *if poſſible*, let ſome polygon ABCFE, whoſe ſides CF, EF are unequal, be the greateſt.



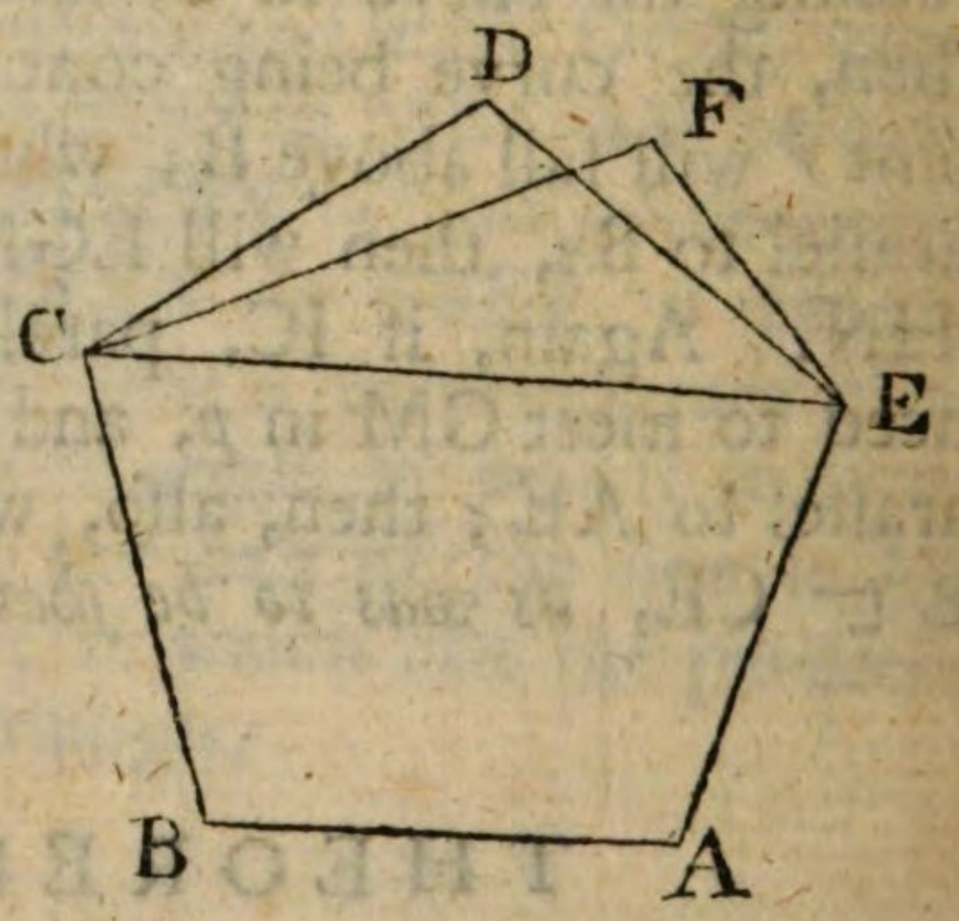
Let CDE be an iſoſceles triangle deſcribed in the ſame ſegment with CFE; which being greater than CFE [r], the whole polygon ABCDE will alſo be greater than the whole polygon ABCFE; *which is repugnant.* Therefore the polygon is the greateſt when the ſides are all equal.

[r] Theor. 6. and 11. 3.

THEOREM X.

Of all right-lined figures, contained under the ſame perimeter, and number of ſides, the greateſt is, when the ſides are all equal.

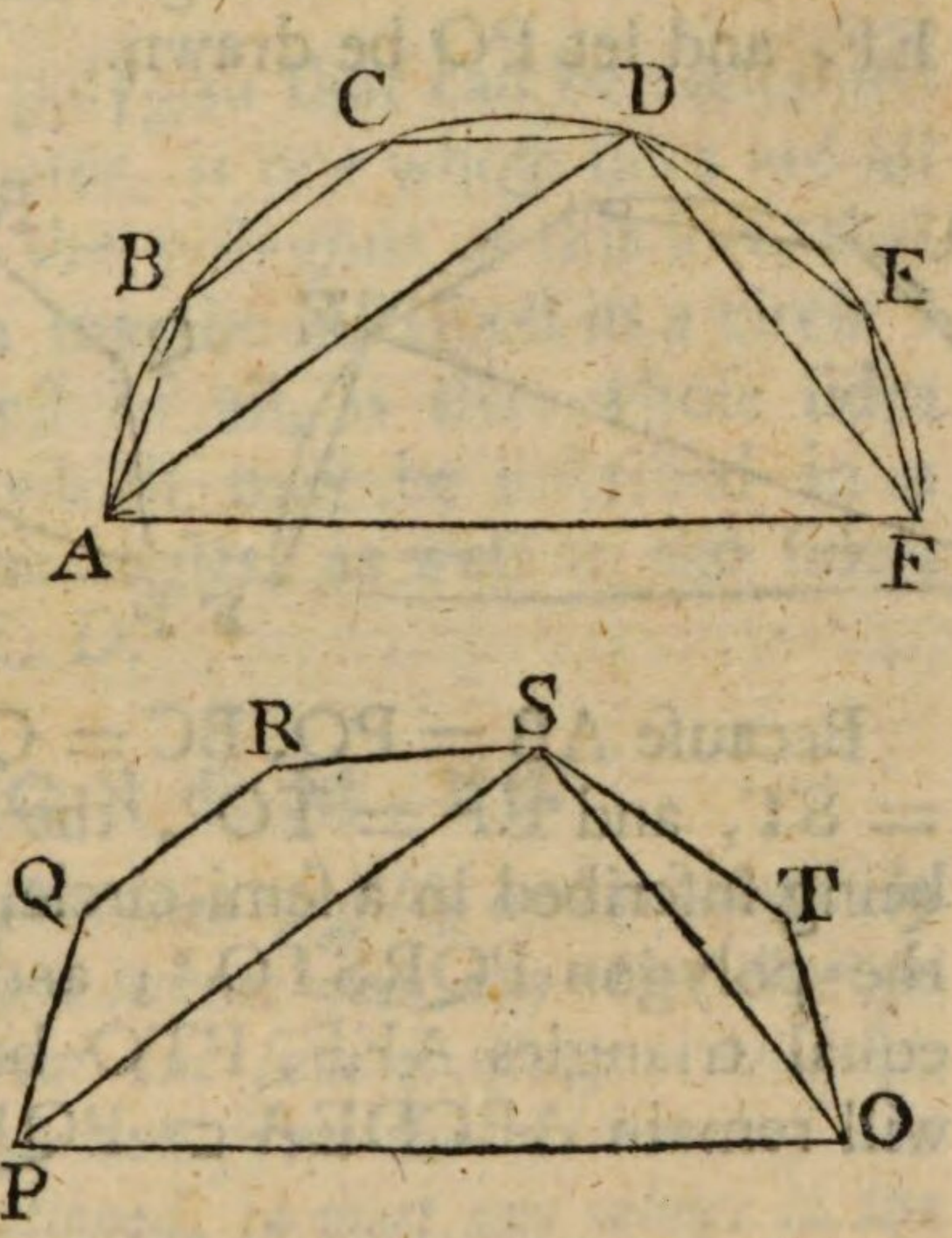
For, if ABCDE be the greateſt poſſible, the triangle CDE muſt, manifeſtly, be greater than any other triangle CFE upon the ſame baſe, whereof the ſum of the other ſides is alſo the ſame. But, *by Theorem* V. the greateſt triangle, when the baſe and the ſum of the ſides are given, is that whoſe ſides are equal: Therefore DC and ED are equal. In the ſame manner it appears that BC = CD, *&c.* *Q. E. D.*



THEO-

THEOREM XI.

If all the ſides of a polygon, except one, be given in length, and their poſition be required, ſo as to make the polygon itſelf the greateſt poſſible; I ſay, their poſition muſt be ſuch, that two lines drawn from the extremes of the unknown ſide to any angle of the polygon, ſhall form a right-angle.



For, if you would have the polygon ABCDEF to be the greateſt poſſible, and yet ADF, ſubtended by the unknown ſide AF, not a right-angle: Then let PSO be a right-angle, contained under $PS = AD$, and $OS = FD$; and upon PS and OS, let the figures PSRQ and OST be deſcribed equilateral, and equal, to ADCB and FDE[s]. [s] 10. 6.

The triangle PSO is greater than ADF[t]; therefore, PSRQ being $=$ ADCB, and $OST = FDE$[u], the whole polygon PQRSTO is alſo greater than the whole polygon ABCDEF[w], *which is repugnant.* [t] Theor. 4. [u] Conſtr. [w] Ax. 6. 1.

COROLLARY.

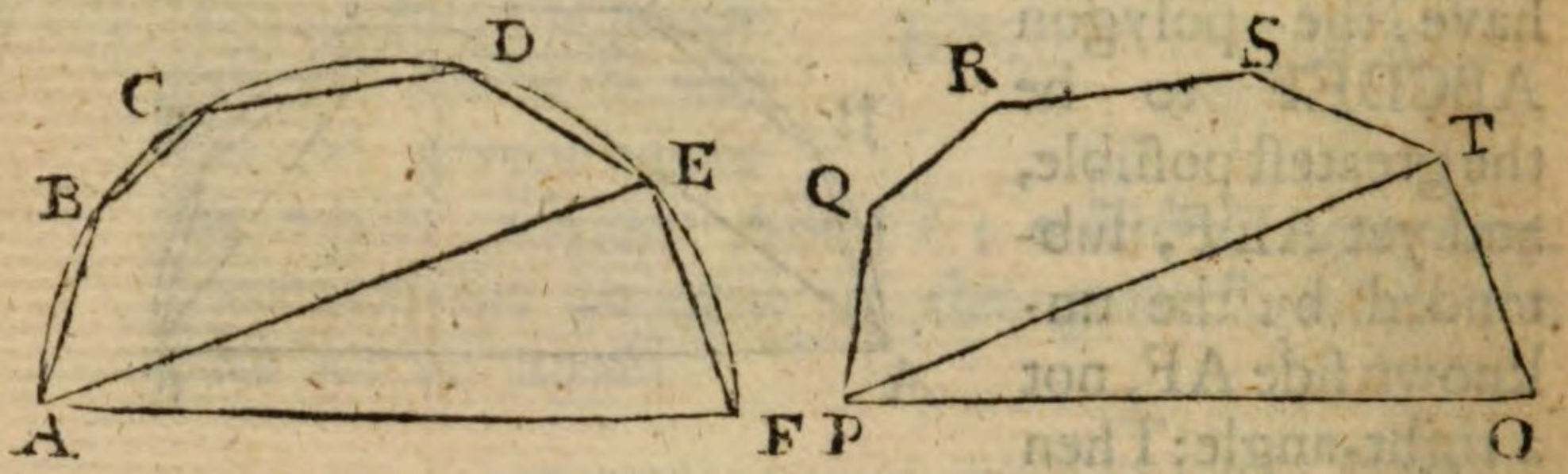
Hence, becauſe the angle in a ſemi-circle is a right-angle[x]; it appears, that the greateſt polygon that can be contained under any propoſed number of given ſides, and one other ſide any how taken, will be, when *it* may be inſcribed in a ſemi-circle, whereof the indetermined line will be the diameter. [x] 13. 3.

THEO.

THEOREM XII.

A polygon ABCDEA *in a circle, is greater than any other polygon* PQRSTP, *whatever, whose ſides are the ſame both in length and number.*

Let AF be the diameter of the circle, and join E, F; alſo make the angle PTO = AEF, TO = EF, and let PO be drawn.

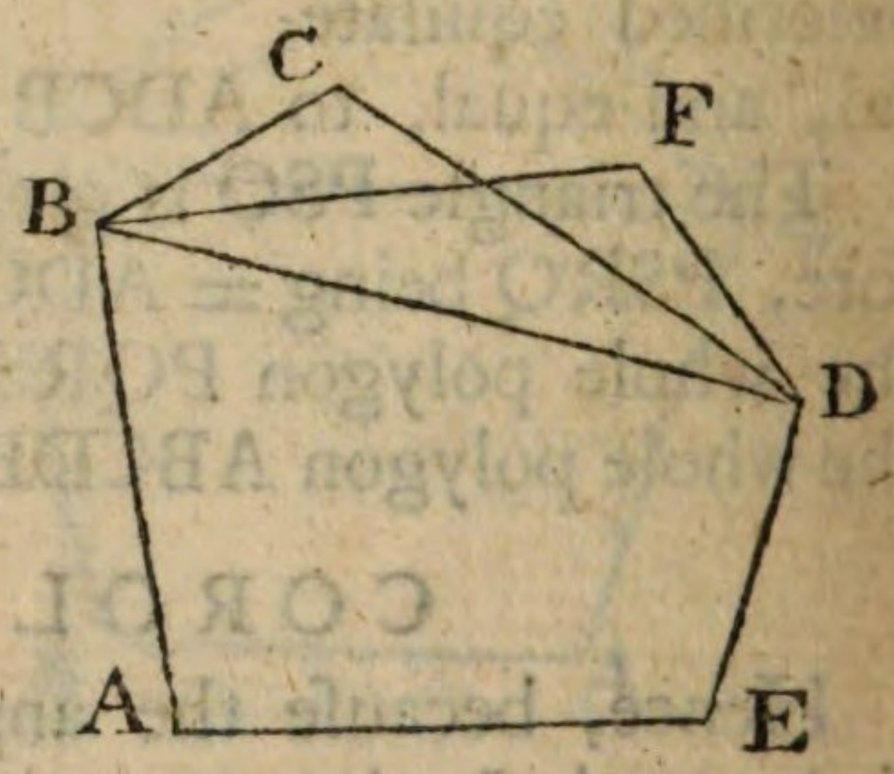


Becauſe AB = PQ, BC = QR, CD = RS, DE = ST, and EF = TO[y], the polygon ABCDEF, being inſcribed in a ſemi-circle, will be greater than the polygon PQRSTO[z]; and, if from theſe, the equal triangles AEF, PTO be taken away, there will remain ABCDEA ⊏ PQRSTP. *Q. E. D.*

[y] Hyp.

[z] Theor. 11.

That the magnitude of the greateſt polygon, which can be contained under any number of unequal ſides, does not at all depend upon the order in which thoſe lines are connected to each other, will appear, thus. Let ABCDE be the greateſt, one way, or according to one order of the ſides; and upon BD let a triangle BDF be conſtituted whoſe ſides DF, and BF are, reſpectively, equal to BC and DC; then, the triangles BCD, BFD being equal, the whole



whole polygons ABCDE and ABFDE will likewise be equal, notwithſtanding their equal ſides BC, DF, *&c.* are placed according to different orders.

THEOREM XIII.

Of all polygons, contained under the ſame perimeter, and number of ſides; that whoſe ſides, and angles, are equal, is the greateſt.

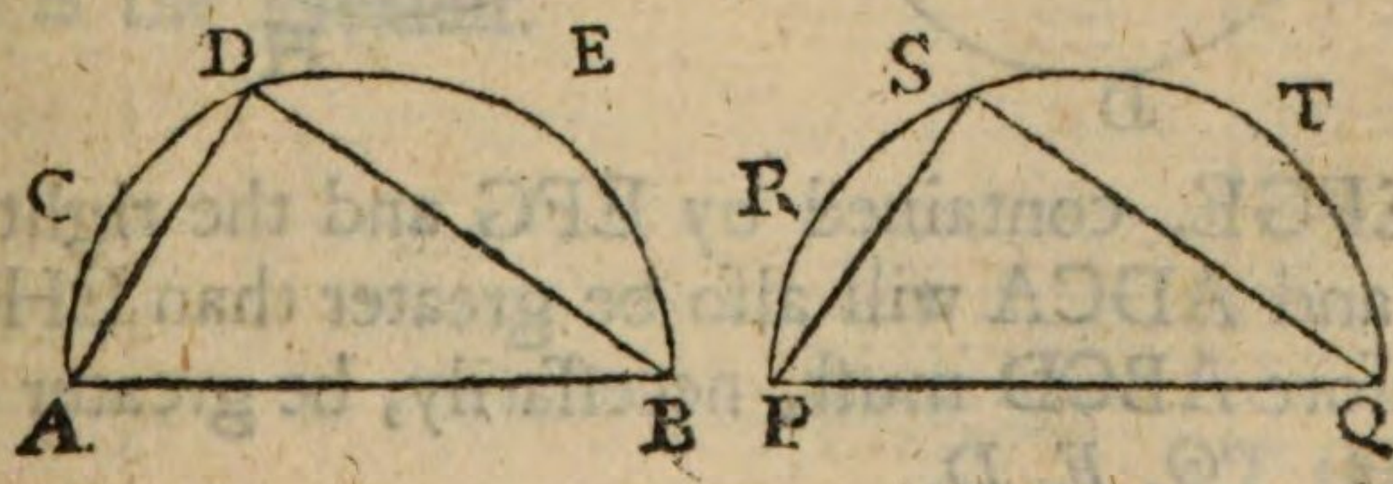
For, the greateſt polygon that can be contained under a given perimeter, is one whoſe ſides are all equal [a]. But, of all the polygons of this ſort, that is the greateſt which may be inſcribed in a circle [b]: Therefore the greateſt of all, is that whoſe ſides are all equal, and which may be inſcribed in a circle, or whereof the angles, as well as the ſides, are all equal. *Q. E. D.*

[a] Theor. 10.
[b] Theor. 12.

THEOREM XIV.

The greateſt area that can poſſibly be contained by one right-line, any how taken, and any other line or lines, whatever, whereof the ſum is given; will be, when two right-lines drawn from the extremes of the unknown line firſt mentioned, to meet any where in the given boundary, make right-angles with each other.

For, if you would have the area ACDEBA, contained by ſome right-line AB, and ACDEB whereof the length is given, to be the greateſt poſſible, and ADB, at the ſame time, not a right-angle: Then, let PSQ be a right-angle, contained under PS = AD, and QS = BD; and, having joined



PQ,

PQ, upon PS and QS conceive two figures PRS and QST to be formed, equal, and alike in all reſpects to ACD and DBE. Since the area PSQ is
[c] Theor 4. greater than ADB [c]; it is manifeſt, that the area PRSTQP, contained by the right-line (PQ) and
[a] Hyp. PRSTQ (= ACDEB [d]) will alſo be greater than
[e] Ax. 6. the area ACDEBA [e], *which is repugnant:* Therefore the area ACDEBA cannot be the greateſt poſſible, unleſs the angle ADB be a right one. *Q. E. D.*

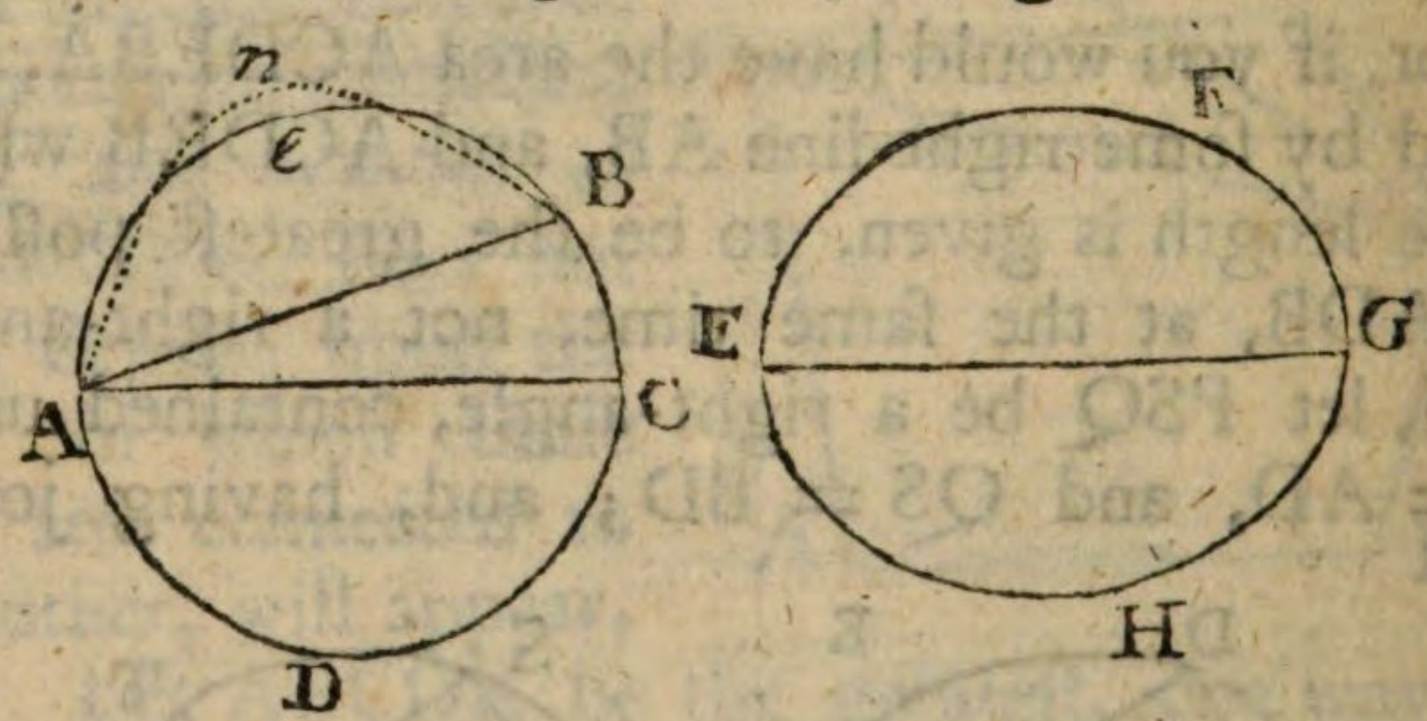
COROLLARY.

Hence, becauſe the angle in a ſemi-circle is a
[f] 13. 3. right-angle [f], it is evident that the area will be the greateſt poſſible, when the given length, or boundary, forms the arch of a ſemi-circle; whereof the indetermined right-line propoſed is the diameter.

THEOREM XV.

Of all plane figures ABCD, EFGH, *contained under equal perimeters (or limits), the circle* (ABCD) *is the greateſt.*

For, if the diameter AC be drawn, and EFG be taken equal to the arch ABC; then the area ABCA will *(by the precedent)* be greater than the



area EFGE, contained by EFG and the right-line EG; and ADCA will alſo be greater than EHGE: Therefore ABCD muſt, neceſſarily, be greater than EFGH. *Q. E. D.*

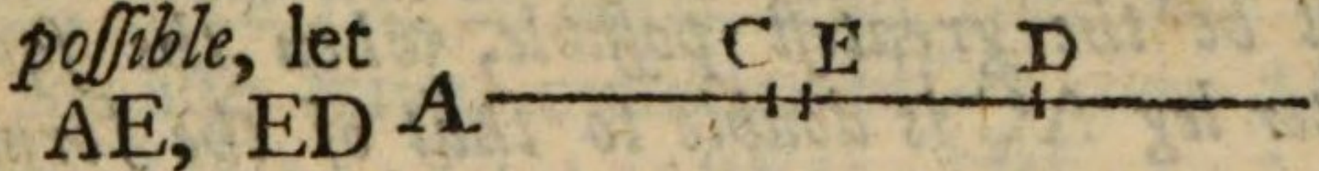
COROL-

COROLLARY.

Hence it appears, that the greateſt area that can poſſibly be contained by a right-line AB, and a curve-line A*e*B, both given in length; will be, when the latter is an arch of a circle. For, let A*n*B be any other curve-line, equal to A*e*B, and let the whole circle A*e*BCD be completed; which will (it is proved) be greater than the mixed figure A*n*BCD; and conſequently, by taking away the common ſegment ABCD, there will remain A*e*BA greater than A*n*BA.

THEOREM XVI.

The greateſt parallelepipedon that can be contained under the three parts of a given line AB, *any how taken, will be when all the parts are equal to each other.*



For, *if poſſible*, let two parts AE, ED be unequal. Biſect AD in C; then will the rectangle under AE (AC + CE) and ED (AC − CE) be leſs than AC^2 (or AC × CD) by the ſquare of CE [g]. Therefore the ſolid AE × ED × DB will alſo be leſs than the ſolid AC × CD × DB [h]; *which is contrary to hypotheſis.*

[g] 7. 2.

[h] 21. 7.

COROLLARY.

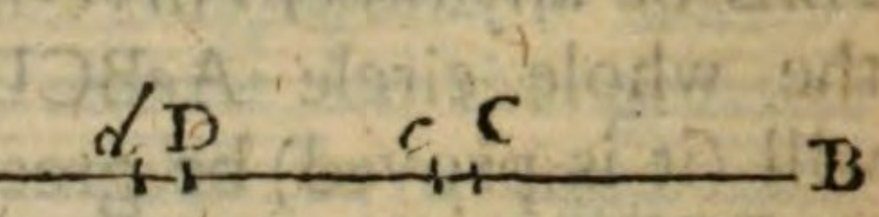
Hence, of all rectangular parallelepipedons, having the ſum of their three dimenſions the ſame, the cube is the greateſt.

THEO-

THEOREM XVII.

The greateſt parallelepipedon $AC^2 \times CB$ *that can poſſibly be contained under the ſquare of one part* AC *of a given line* AB *and the other part* AC, *any how taken; will be, when the former part is the double of the latter.*

For, let A*c* and B*c* be any other parts, into which the given line AB may be divided; and let AC and A*c* be biſected in D and *d*. So ſhall $AC^2 \times CB =$ $4AD \times DC \times CB$ [i] $\sqsubset$ $4Ad \times dc \times cB$ [k] $(Ac^2 \times cB$ [i]) *by the precedent.* *Q. E. D.*



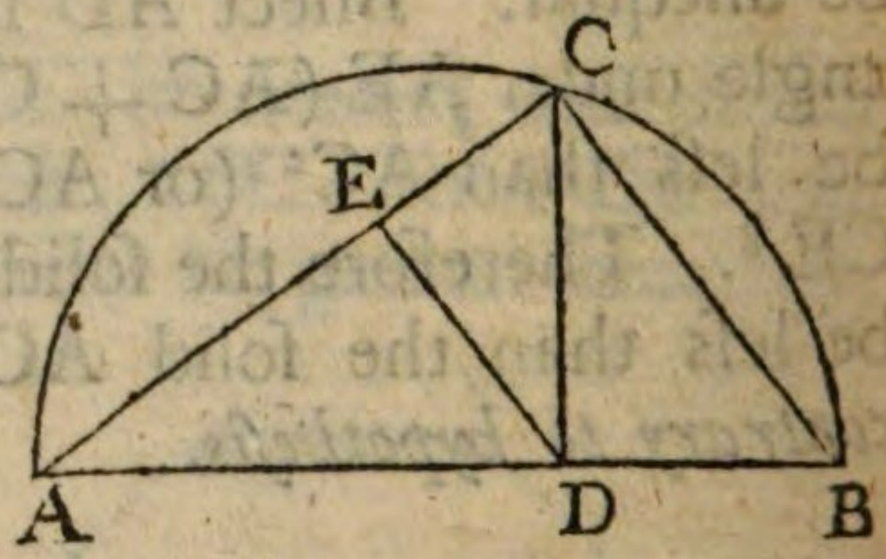
[i] Cor. to 6. 2. and 7.

[k] Cor. to 6. 2.

THEOREM XVIII.

The hypothenuſe AB *of a right-angled triangle* ABC *being given; the ſolid* $BC \times AC^2$ *contained under one leg* BC *and the ſquare of the other* AC, *will be the greateſt poſſible, when the ſquare of the latter leg* AC *is double to that of the former* BC.

For, if CD be conceived perpendicular to AB, and DE to AC; it will be $AC^2(AB \times AD$ [l]$)$: $AB^2 :: AD : AB$ [m] $:: DE : BC$ [n]; and conſequently $AC^2 \times BC = AB^2 \times DE$ [o]; which (as AB^2 is given) will, evidently, be the greateſt poſſible, when DE, or its ſquare [p] (DE^2) is the greateſt poſſible. But $DE^2 : AD^2 ::$ [q] $BC^2(BD \times AB$ [l]$) : AB^2 :: BD : AB$ [m]; and therefore $DE^2 \times AB = AD^2 \times BD$ [o]; which (and conſequently DE^2) will be the greateſt poſſible, when AD is the double of BD [r]; that is, when AC^2 $(AD \times AB)$ is the double of BC^2 $(BD \times AB)$. *Q. E. D.*



[l] Cor. to 19. 4.

[m] 21. 7.

[n] 14. 4.

[o] 23. 7.

[p] Cor. 2. to 6. 2.

[q] Cor. to 11. 4.

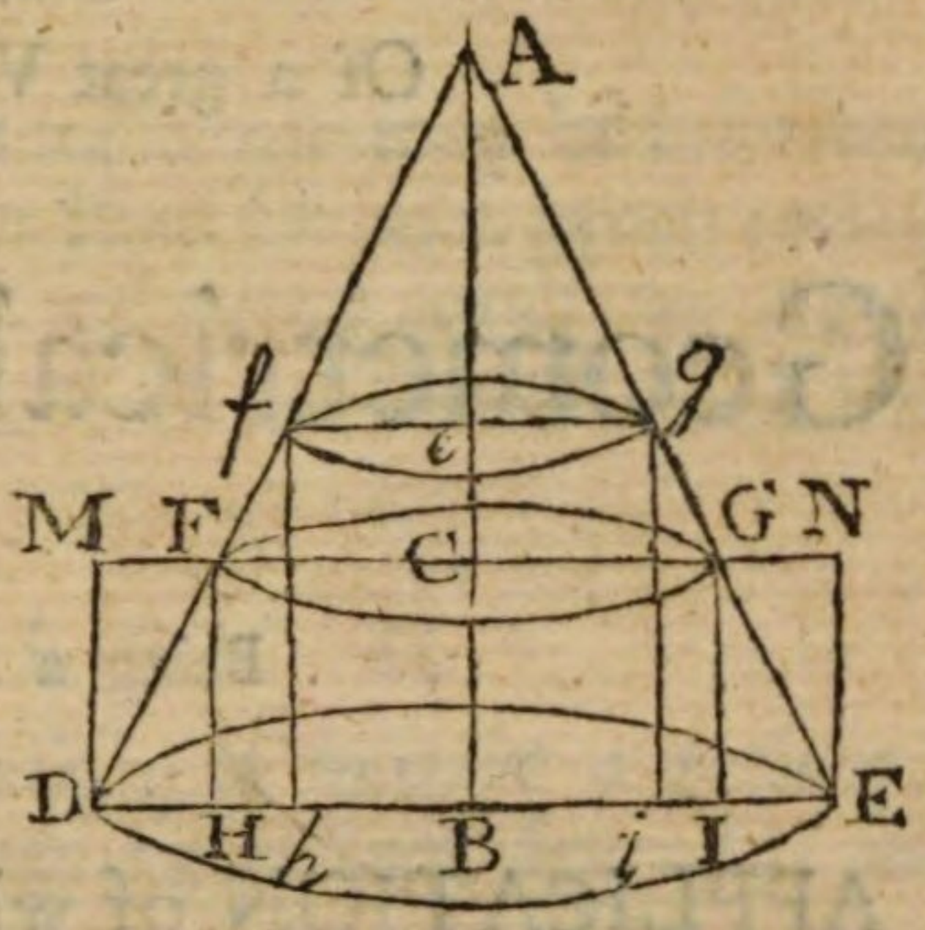
[r] Theor. 17.

THEO-

THEOREM XIX.

The altitude BC *of the greateſt cylinder* HG *that can poſſibly be inſcribed in any cone* ADE, *is one-third part of the altitude* AB *of the cone, and the cylinder itſelf* $\frac{4}{9}$ *parts of the cone.*

For, let *gh* be any other cylinder inſcribed in the cone; and it will be, $AC^2 \times BC : CG^2 \times BC :: AC^2 : {}^{s}CG^2 :: Ac^2 : cg^{t} :: Ac^2 \times Bc : {}^{s}cg^2 \times Bc$; whence, *by alternation*, $AC^2 \times BC : Ac^2 \times Bc :: CG^2 \times BC : cg^2 \times Bc$: and ſo likewiſe is the cylinder HG to the cylinder ${}^{u}hg$; but $AC^2 \times BC$ is greater than $Ac^2 \times Bc$; therefore HG is alſo greater than *hg*. Again, ſince $AC = \frac{2}{3}AB$, and therefore $CG = \frac{2}{3}BE$; we alſo have, cylinder HG : cone ADE (or cylinder ${}^{y}DN$) :: CG^2 ($\frac{4}{9}BE^2$) : ${}^{z}BE^2 :: \frac{4}{9} : 1$. *Q. E. D.*



s 22. 7.
t Cor. 11. 4.
u 3. and 5. of 8.
w Theor. 17.
x Ax. 4.
y Cor. 3. to 8. 8.
z 21. 7. and 3. 8.

SCHOLIUM.

From this propoſition, by reaſoning as in the Scholium to Theorem VIII. it will appear, that the leaſt cone that can be deſcribed about, and the greateſt cylinder that can poſſibly be deſcribed in, any ſolid generated by the rotation of a curve, concave to its axis, will be, when the ſub-tangent is two-thirds of the altitude of the cone, or twice the altitude of the cylinder; and that the two figures will be in the ratio of nine to four. From whence the dimenſions of the greateſt and leaſt cylinders and cones, that can be deſcribed in, and about ſolids generated by curves, to which the method of drawing tangents is known, may be readily determined.

THE

CONSTRUCTION

Of a great VARIETY of

Geometrical Problems.

Being a farther

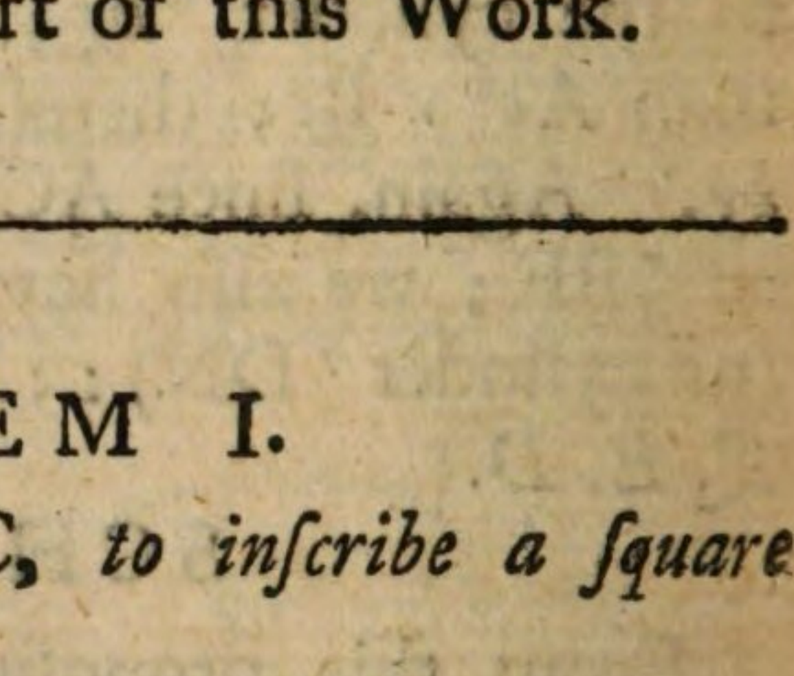
APPLICATION of what has been delivered in the Elementary Part of this Work.

PROBLEM I.

In a given triangle ABC, *to inſcribe a ſquare*

DEFN.

CONSTRUCTION.



FROM any point M, in either ſide, upon the baſe AB, let fall the perpendicular MG; make MR perpendicular, and equal, thereto, and let ARE be drawn, meeting the other ſide of the triangle in E; then draw ED parallel, and EF and DN perpendicular, to AB; *and the thing is done.*

DE-

DEMONSTRATION.

Let RS be drawn parallel to EF: Then *(by ſimilar triangles)* RS (MG) : EF (: : AR : AE) : : MR : DE: Therefore, as MG and MR are equal, *by Conſtruction*, EF and DE will likewiſe be equal.

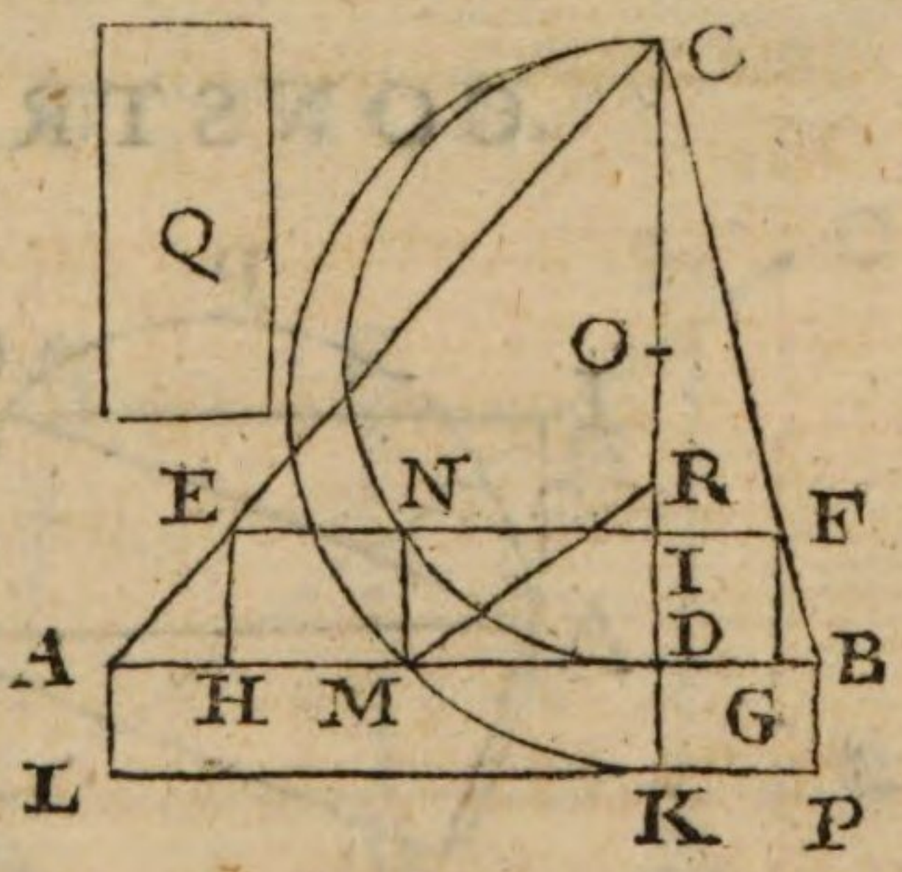
By the ſame method a rectangle may be inſcribed in a triangle, whoſe ſides ſhall be in a given ratio; if MR and MG (inſtead of being equal) be taken in the given ratio; the reſt of the Conſtruction being exactly the ſame.

PROBLEM II.

In a given triangle ABC, *to inſcribe a rectangle* EFGH *equal to any given right-lined figure* Q, *not exceeding half the triangle.*

CONSTRUCTION.

On the baſe AB *(by* 7. 6.) let a rectangle ABPL be conſtituted = Q; and let LP meet the perpendicular CD of the triangle (produced) in K. Then *(by* 17. 5.) let CD be divided in I, ſo that CI × DI = CD × DK (that is, let two ſemi-circles be deſcribed on CD and CK; drawing MN and NI parallel to CD and AB): So ſhall DI be the altitude of the required rectangle.



DEMONSTRATION.

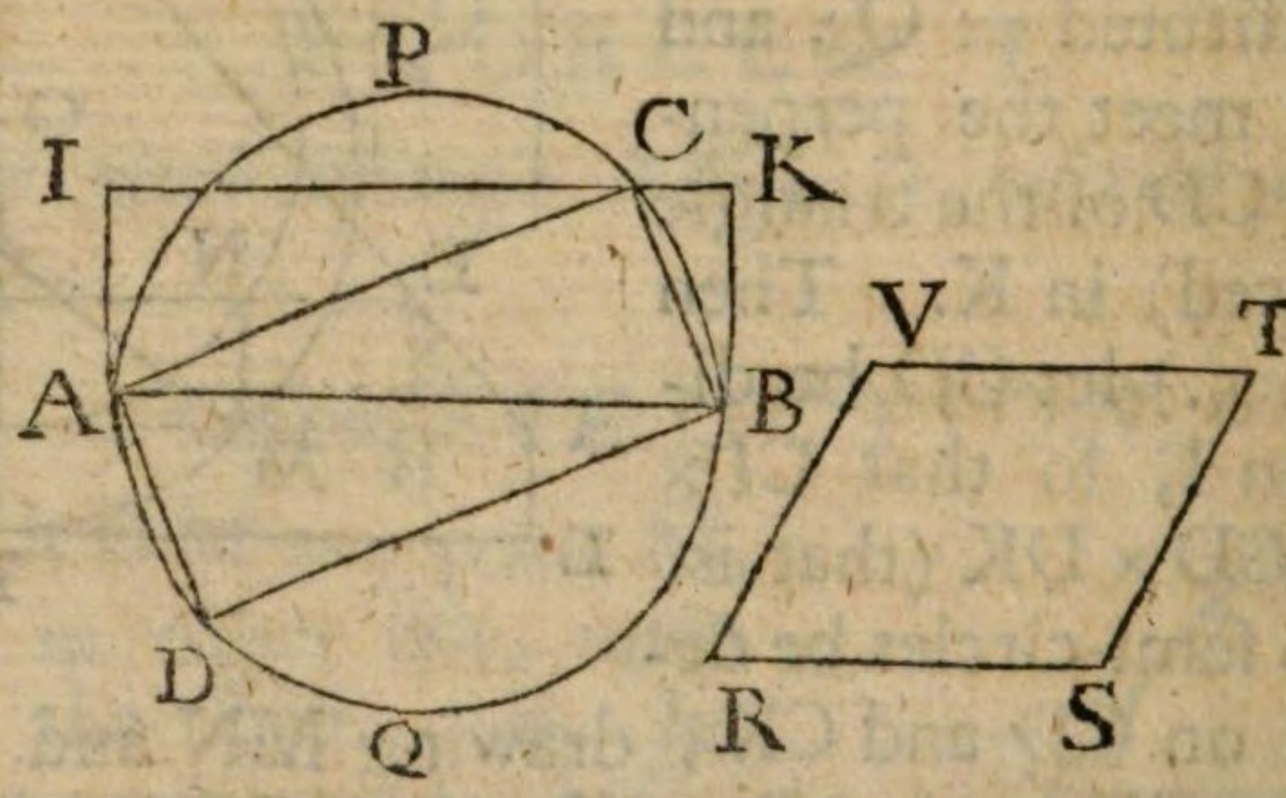
Since *(by Conſtr.)* CI × DI (= NI² = MD²) = CD × DK, thence will DI : DK : : CD : CI : : AB : EF *(by* 20. 5.); and conſequently DI × EF *(by* 10. 4.) = AB × DK = Q. *Q. E. D.*

That the Problem will be impoſſible, when Q is greater than half the triangle, is evident from the Conſtruction, as well as from the Theorem on *p.* 200. It may alſo be obſerved, that there is another way, beſides that uſed above, for dividing CD in the manner propoſed; which (though not more obvious) is, in point of conciſeneſs, rather preferable; and is thus. Having *(as before)* found a mean-proportional DM between CD and DK, and biſected CD in O; from M to CD apply MR = OD, and take OI = RD. So ſhall CI × DI = OD^2 — OI^2 *(by* 7. 2.) = MR^2 — RD^2 *(by Hyp.)* = DM^2 = CD × DK *(as before)*.

PROBLEM III.

In a given circle APBQ, *to inſcribe a rectangle equal to a given right-lined figure* RSTU, *not exceeding half the ſquare of the diameter.*

CONSTRUCTION.



Upon the diameter AB deſcribe the rectangle ABKI = RSTU *(by* 7. 6.); and from the point C, where the ſide KI interſects the periphery of the circle, draw CA and CB, parallel to which draw BD and AD; then will ACBD be the rectangle that was to be conſtructed.

DE-

DEMONSTRATION.

The lines AC, BD, and AD, BC being parallel *(by Conſtr.)* and the angle ACB a right-one *(by* 13. 3.) the figure ACBD is a rectangle *(by Cor. to* 24. 1.) and D is alſo in the circumference of the circle. But ACBD = 2ACB = ABKI = RSTU.

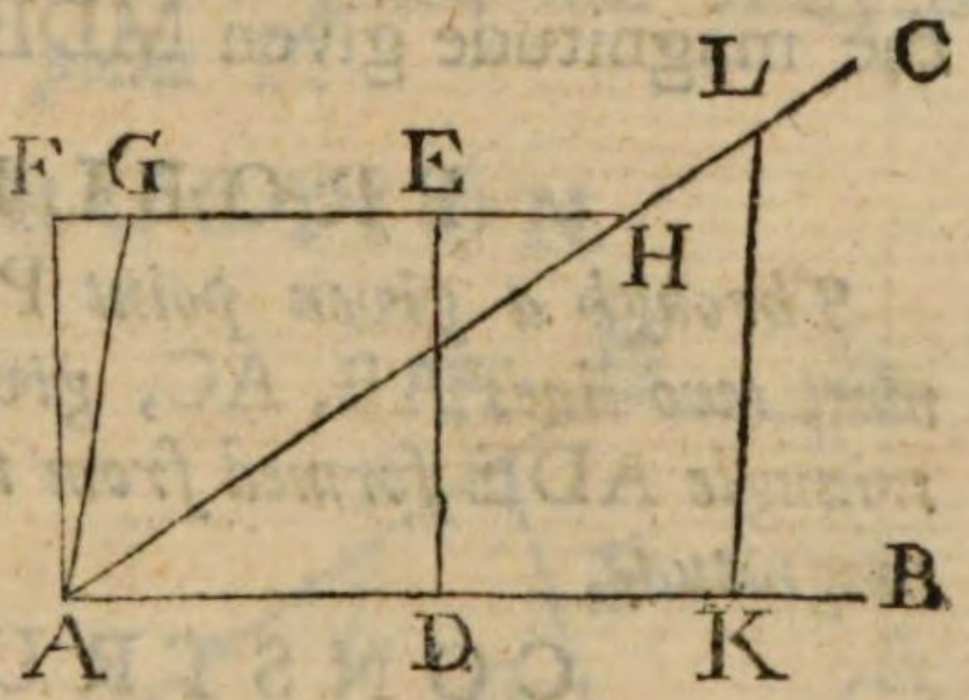
That the Problem will be impoſſible, when BK is greater than $\frac{1}{2}$AB, or when BI (RT) is greater than $\frac{1}{2}$AB^2, is manifeſt from hence, becauſe KI will then fall intirely above the circle.

PROBLEM IV.

To draw a line KL *parallel to a given line* AG, *which ſhall terminate in two other lines* AB, AC, *given by poſition, ſo as to form with them a triangle* AKL, *equal to a given rectangle* ADEF.

CONSTRUCTION.

Let FE, produced, meet AG and AC, in G and H; and, in AB, take a mean-proportional AK between GH and 2EF; then draw KL parallel to AG, *and the thing is done.*



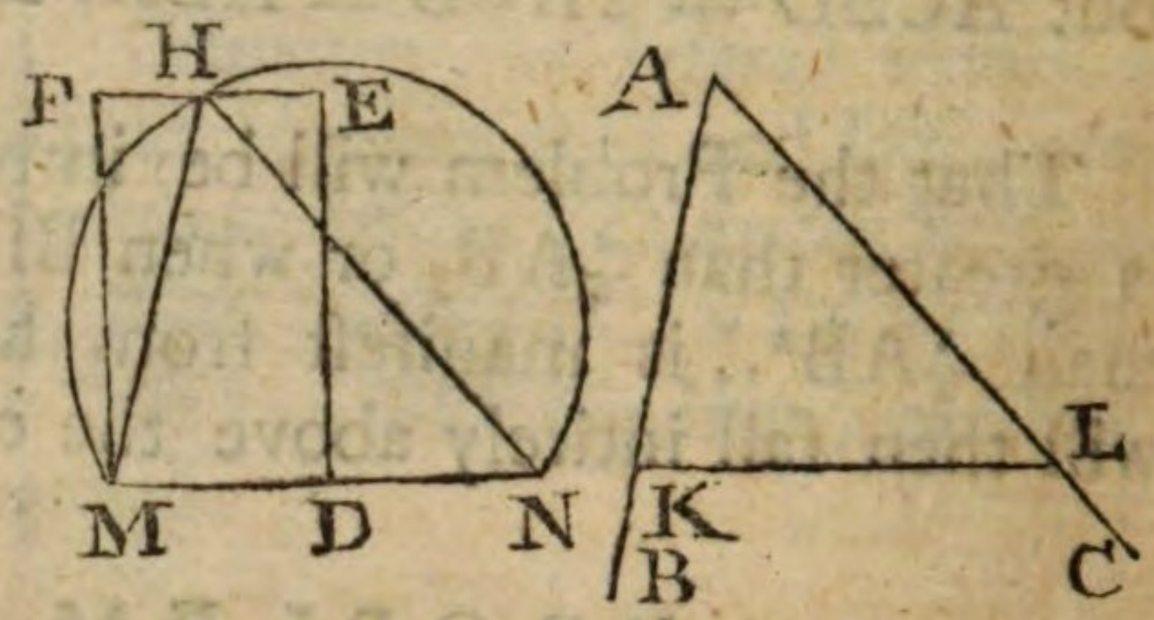
DEMONSTRATION.

The triangles AKL, HGA being equiangular, it will be AKL : HGA : : AK^2 (= GH × 2EF, *by Conſtr.)* : GH^2 : : EF : $\frac{1}{2}$GH *(by* 7. 4.) : : EF × AF : $\frac{1}{2}$GH × AF (= HGA) : Therefore, the conſequents being equal, the antecedents AKL and EF × AF muſt alſo be equal. *Q. E. D.*

PROBLEM V.

Between two lines AB, AC, *given by position, to apply a line* KL, *equal to a given line* MN, *so that the triangle* AKL *formed from thence, shall be of a given magnitude.*

CONSTRUCTION.

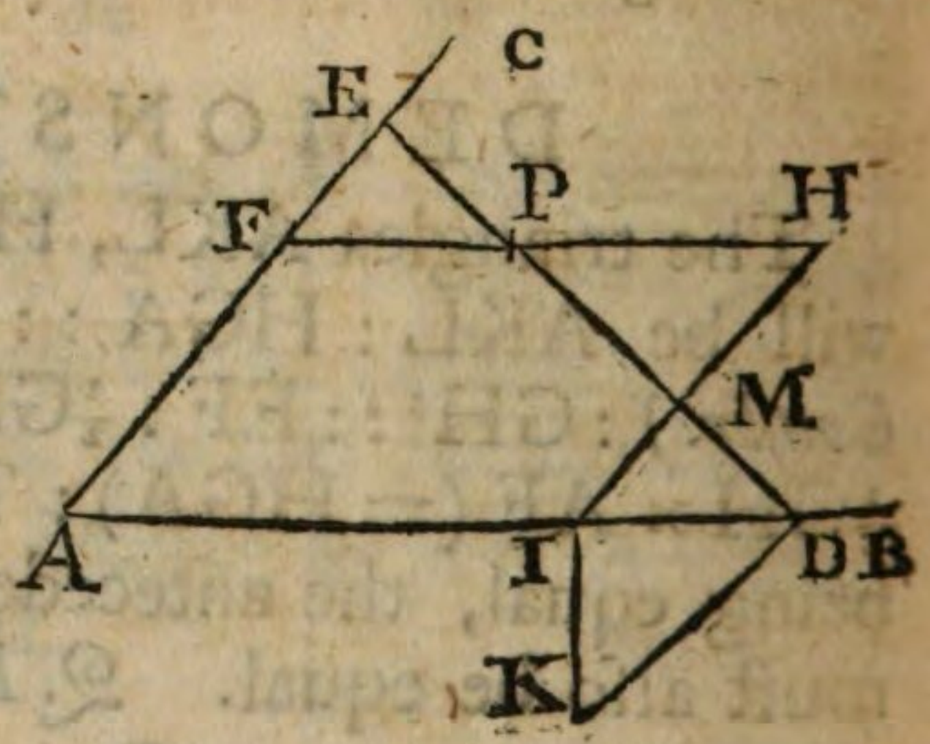


Having bisected MN in D, on MD describe a rectangle MDEF (*by* 7. 6.) = the magnitude given: Also on MN let a segment of a circle be described (*by* 22. 5.) to contain an angle = A; and from its intersection with EF, draw HM and HN; then make AK = HM, AL = HN, and join K, L. So shall the base KL be also = the base MN, and the triangle AKL equal, and like in all respects, to HMN; which last (*by Cor. to* 2. 2.) is, manifestly, equal to the magnitude given MDEF. *Q. E. D.*

PROBLEM VI.

Through a given point P, *to draw a line* EPD *to meet two lines* AB, AC, *given by position, so that the triangle* ADE *formed from thence, shall be of a given magnitude.*

CONSTRUCTION.



Draw FPH parallel to AB, intersecting AC in F; and on AF, let a parallelogram AFHI be formed, equal to the given area of the triangle: Make IK perpendicular to AI, and equal

to

to FP; and from K, to AB, apply KD = PH; then draw DPE, *and the thing is done.*

DEMONSTRATION.

The triangles PHM, PFE and MDI, by reason of the parallel lines, are similar; and therefore, since the three homologous sides PH (KD), FP (IK) and DI are such, as to form a right-angled triangle (*by Constr.*), the triangle PHM on the first of them, is equal to both the other two FPE and MDI (*by* 29. 4.): and, if to these equal quantities, AFPMI be added; then will AFHI be also = ADE.

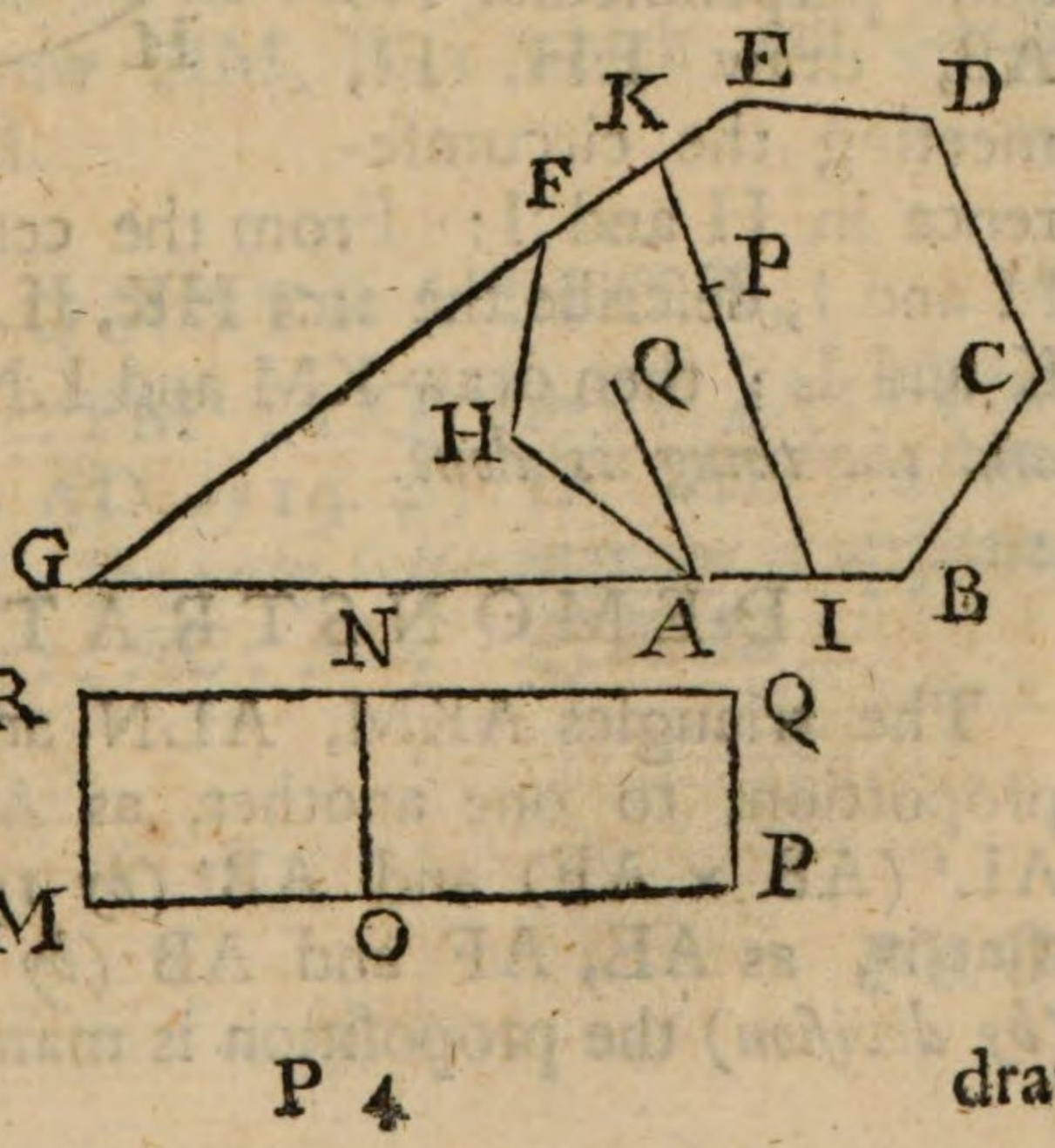
This Problem will be impossible, when KD (PH) is less than KI (PF); that is, when the area given is less than a parallelogram under AF and 2FP.

PROBLEM VII.

From a given polygon ABCDEFH, *to cut off a part* AIKFH, *equal to a given rectangle* MN, *by a line* (IK), *either parallel to a given line* AQ, *or passing through a given point* P.

CONSTRUCTION.

Let BA and EF be produced to meet in G; and upon ON let a rectangle OQ be constituted (*by* 7. 6.) equal to AGFH; then, by Prob. the 4th, or 6th, according to the case proposed,



draw

draw IK, ſo as to make the triangle GKI = MQ, *and the thing is done.*

The Demonſtration whereof is manifeſt from the Conſtruction.

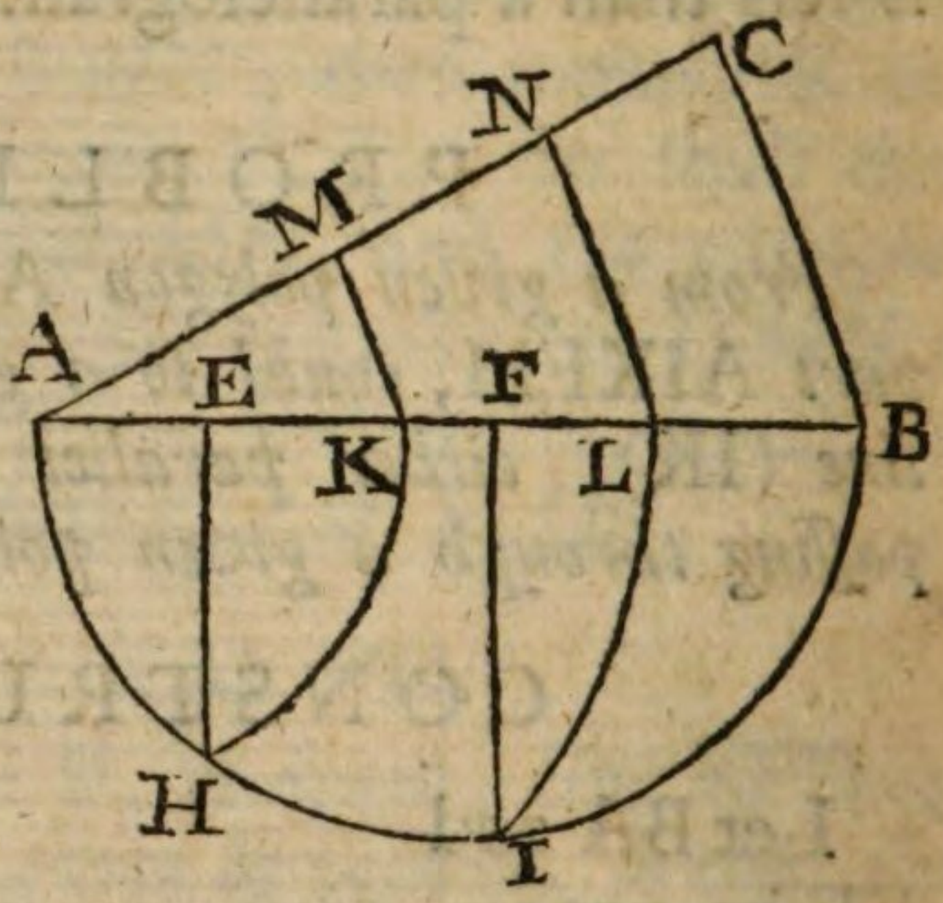
And, in the ſame manner, the polygon may be divided according to any given ratio; becauſe, the whole being given, each part will be given.

PROBLEM VIII.

To divide a given triangle ABC *into any propoſed number of parts* (AKM, KN, LC) *ſo as to have any given proportion to each other; by means of lines drawn parallel to one of the ſides* BC *of the triangle.*

CONSTRUCTION.

Let AB be divided into parts, AE, EF, FB, having the ſame given proportion to each other, as the parts of the triangle are to have. Upon AB let a ſemi-circle AHIB be deſcribed; and perpendicular to AB, draw EH, FI, meeting the circumference in H and I: From the center A, through H and I, deſcribe the arcs HK, IL, meeting AB in K and L; then draw KM and LN parallel to BC, *and the thing is done.*



DEMONSTRATION.

The triangles AKM, ALN and ABC, are in proportion, to one another, as AK^2 (AB × AE), AL^2 (AB × AF) and AB^2 (*by* 19. and 24. 4.); that is, as AE, AF and AB (*by* 7. 4.) Whence (*by diviſion*) the propoſition is manifeſt.

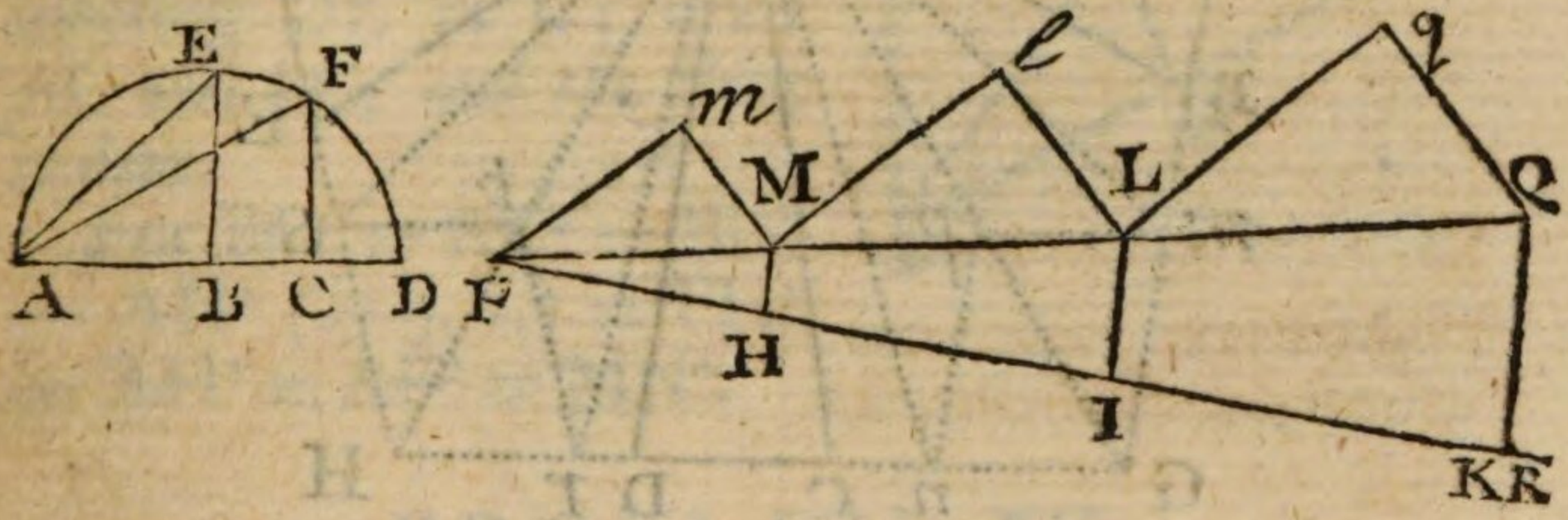
PRO-

PROBLEM IX.

To divide a given line PQ *into any proposed number of parts, so that similar right-lined figures* PMm, MLl, LQq *described upon them, shall have the same given ratio among themselves, as an equal number of right-lines* AB, AC, AD *assigned.*

CONSTRUCTION.

Upon the greatest AD of the given lines AD, AC, AB, describe a semi-circle AEFD; and perpendicular to AD, draw BE and CF, meeting the circumference in E and F; and, having drawn



PR, at pleasure, in it take PH = dist. AE, HI = dist. AF and IK = AD; draw KQ, and parallel thereto draw HM, IL; which will divide PQ, as required.

DEMONSTRATION.

PMm : LQq : : PM^2 : LQ^2 (*by* 26. 4.) : : PH^2 (= AE^2 = AB × AD, *by* 19. 4.) : IK^2 (AD^2) : : AB : AD (*by* 7. 6.) In the very same manner it appears, that MLl : LQq : : AC : AD. *Q. E. D.*

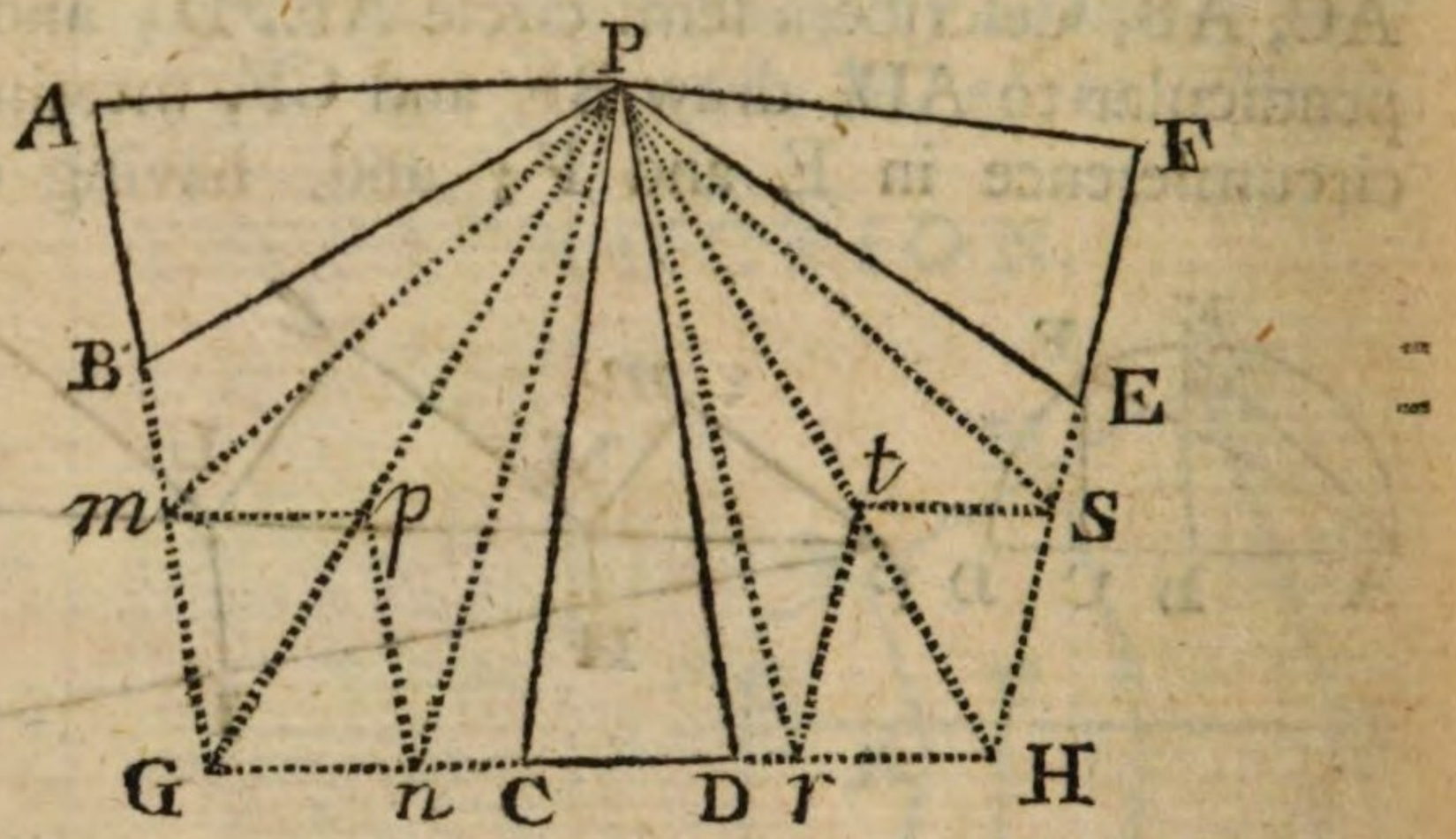
PRO-

PROBLEM X.

To determine the position of a point P, *so that lines drawn from thence to the extremes of three right-lines* AB, CD, EF, *given in length and position, shall form three triangles* ABP, CPD, EPF, *mutually equal to each other.*

CONSTRUCTION.

Let the given lines be produced to meet in G and H; in which take G*m* = AB, H*s* = EF, and



G*n*, H*r*, equal each to CD: Complete the parallelograms G*mpn*, H*rts*; and let the diagonals G*p* and H*t* be produced till they meet in P, *and the thing is done.*

DEMONSTRATION.

Let PA, PB, P*m*, &c. be drawn. The triangles GP*n*, GP*m*, having the same base GP, and equal altitudes (because GM*pn* is a parallelogram) are therefore equal to each other: But CPD = GP*n*, and APB = GP*m* (*by Cor. to* 2. 2.); whence CPD and APB are likewise equal. And, from the very same reasoning, it appears that CPD and FPE are equal. *Q. E. D.*

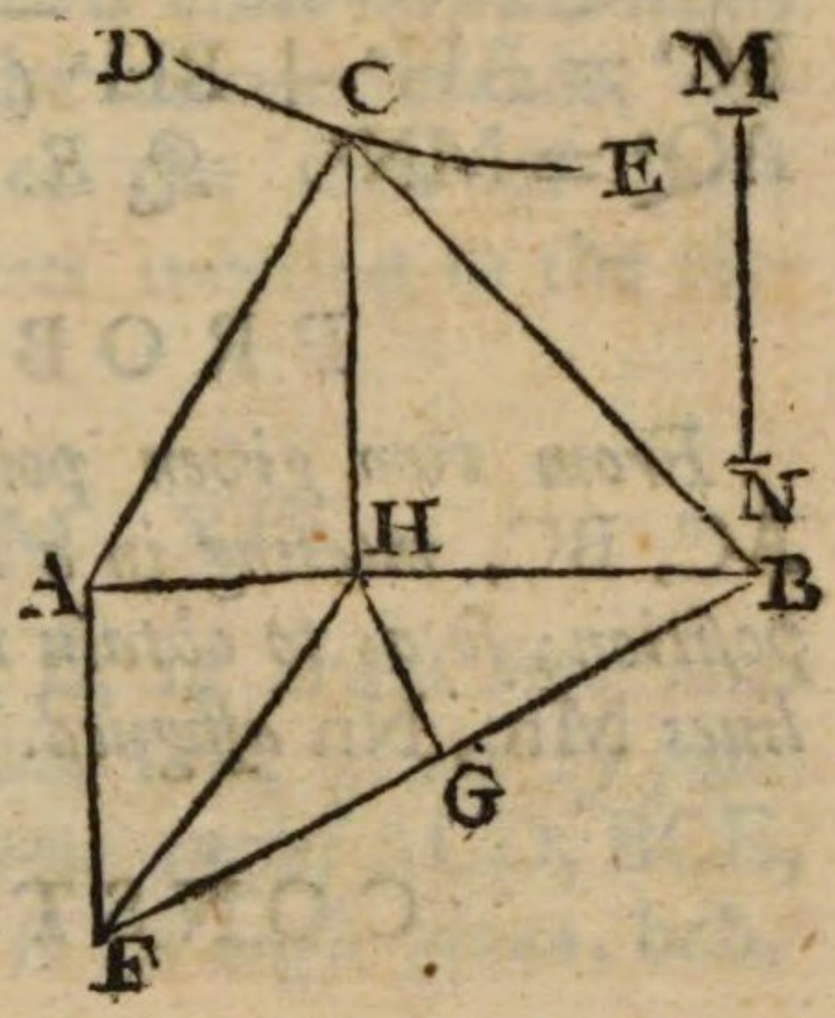
PRO-

PROBLEM XI.

From two given points A, B, *to draw two lines* AC, BC, *to meet in a line* DE *of any kind, given by position; so that the difference of their squares shall be equal to a square given* (MN^2).

CONSTRUCTION.

Make AF perpendicular to the line AB, and equal to MN; draw BF, which biſect with the perpendicular GH; and from its interſection with AB, draw HC perpendicular to AB, meeting DE in C; draw AC and BC, *and the thing is done.*



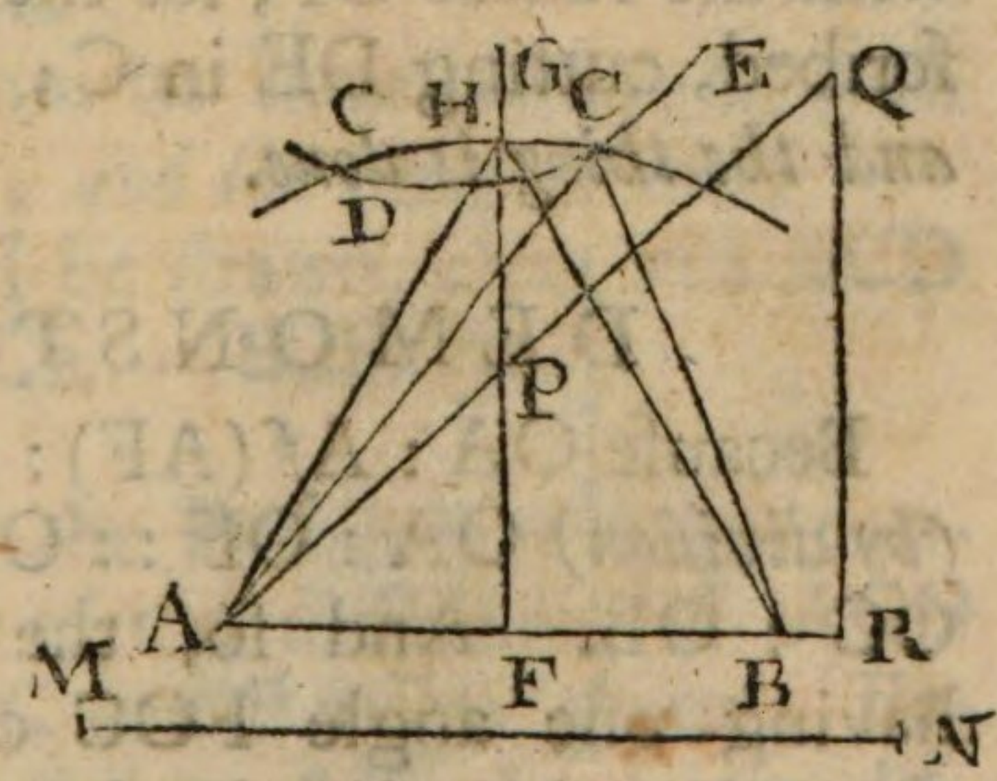
For $BC^2 - AC^2 = BH^2$
$- AH^2$ (*by* 9. 2.) $= FH^2$
$- AH^2 = AF^2 = MN^2$.

PROBLEM XII.

From two given points A, B, *to draw two lines* AC, BC, *to meet in a line* DE *of any kind, given by position; ſo that the ſum of their ſquares ſhall be equal to a given ſquare,* MN^2.

CONSTRUCTION.

Biſect AB with the perpendicular FG, in which take FP = FA, and draw APQ = MN; on AB (produced, if neceſſary) let fall the perpendicular QR; from A to FG apply AH = AR, and



about

about the center F, thro' H, let the circumference of a circle be deſcribed; and from its interſection C with the given line DE, draw CA and CB, *and the thing is done.*

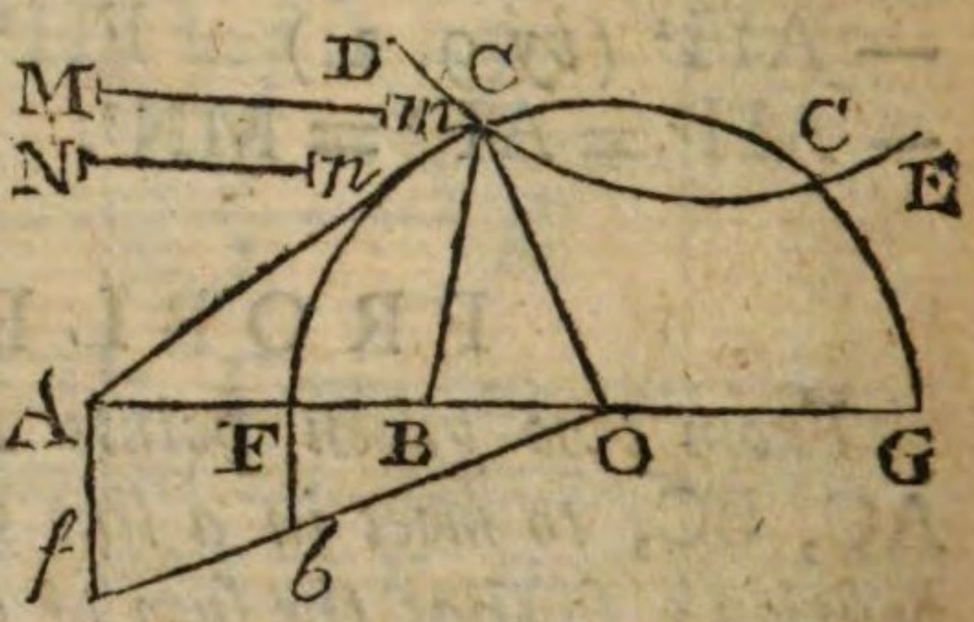
DEMONSTRATION.

Let BH be drawn; which being = AH = AR = RQ (becauſe FP = AF); thence will $AC^2 + BC^2 = AH^2 + BH^2$ (*by* 20. 3.) $= AR^2 + RQ^2 = AQ^2 = MN^2$. *Q. E. D.*

PROBLEM XIII.

From two given points A, B, *to draw two lines* AC, BC, *meeting in a line* DE *of any kind, given by poſition; ſo as to obtain the ratio of two unequal right-lines* Mm, Nn *aſſigned.*

CONSTRUCTION.



Having joined the given points, divide AB in F (*by* 15. 5.) ſo that AF : BF : : M*m* : N*n*; make A*f* and F*b* parallel to each other, taking the former = AF, and the latter = FB; and thro' their extremes draw *fb*O, meeting AB, produced, in O; from whence, with the radius OF, let the ſemi-circle FCG be deſcribed, cutting DE in C; then draw AC and BC, *and the thing is done.*

DEMONSTRATION.

Becauſe OA : A*f* (AF) : : OF : F*b* (FB), we have (*by diviſion*) OA : OF : : OF : OB; or OA : OC : : OC : OB. And ſo, the triangles OAC, OCB, having one angle FOC common, and the ſides about it proportional, they muſt, therefore, be ſi-

milar

milar (*by* 15. 4.); whence the other ſides will alſo be proportional, or AC : BC : : OA : OC (OF) : : *Af* : F*b* : : M*m* : N*n* (*by Conſtr.*) *Q. E. D.*

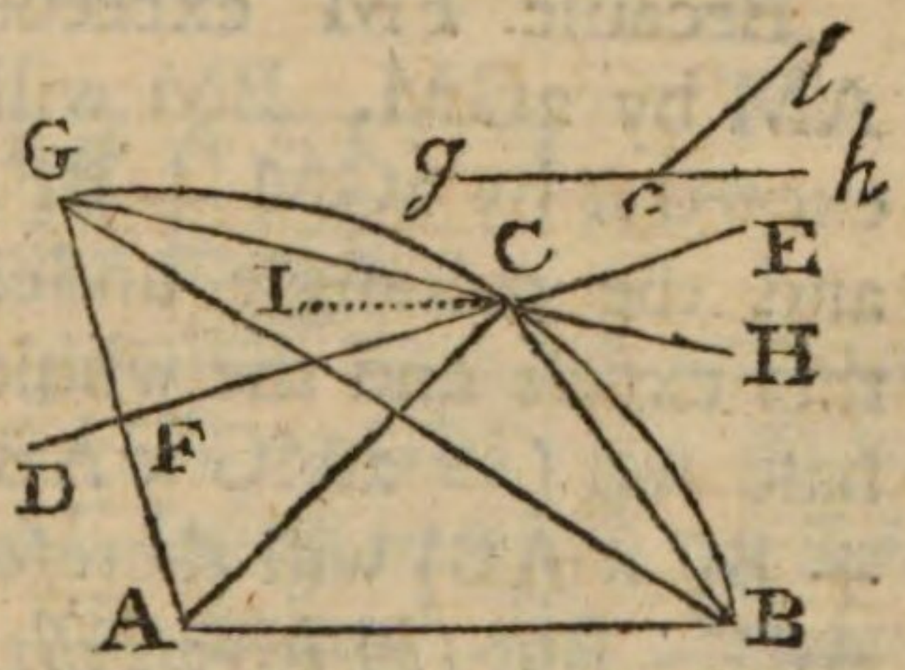
Note, When, in either of the two preceding Problems, the circle deſcribed, neither cuts nor touches the given line DE, the thing propoſed to be done, will be impoſſible; as no two lines drawn from A and B, to meet above the circumference, can poſſibly have their ratio, or the ſum of their ſquares, the ſame as two lines meeting in the circumference.

PROBLEM XIV.

From two given points A, B, *to draw two lines* AC, BC, *meeting in a right-line* DE, *given by poſition; ſo as to make therewith two angles* ACD, BCE, *whoſe difference ſhall be equal to an angle given*, bch.

CONSTRUCTION.

Make AFG perpendicular to DE, and FG = AG; and, having drawn GB, on it let a ſegment of a circle GCB be deſcribed (*by* 22. 5.) to contain an angle equal to the ſupplement (*bcg*) of the given one *bch*; and from its interſection (C) with DE, draw CA and CB; *and the thing is done.*



For, if BC and GCH be drawn, then will ACD = GCD = ECH = BCE — BCH (*bch*).

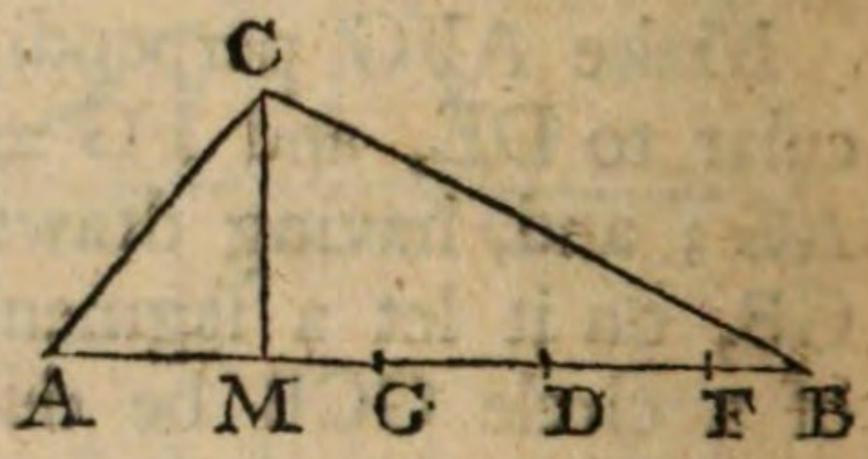
In the ſame manner, the Problem will be conſtructed, when, inſtead of the difference of ACD and BCE, *that* of ABC and BAC is given: Becauſe, when DE is parallel to AB, the latter difference

ference is equal to the former; and, in all other caſes, differs from it by twice the given angle GCI, expreſſing the inclination of the ſaid lines.—When the ſum of the angles ACD and BCE is given, the angle ACB is alſo given: And here, nothing more is neceſſary, than barely to deſcribe, upon AB, a ſegment of a circle to contain the ſaid given angle ACB.

L E M M A.

If, of any three proportional lines AB, DB, FB, *the difference* AF *of the two extremes be biſected in* G; *and if on the greateſt* AB, *as a baſe, a triangle* ABC *be ſo formed, that its leſſer ſide* AC *ſhall be to the diſtance* MG *of the perpendicular from the biſecting point* G, *in the given ratio of* AB *to* DB; *then ſhall the greater ſide* BC *exceed the leſſer* AC *by the given line* DB.

DEMONSTRATION.



Becauſe FM exceeds AM by 2GM, BM will exceed *it* by 2GM + BF; and the rectangle under this exceſs and the whole baſe AB ($= 2MG \times AB + BF \times AB$) will therefore (*by* 9. 2.) be $= BC^2 - AC^2$. But (*by hypotheſis and* 10. 4.) $MG \times AB = AC \times BD$, and $BF \times AB = BD^2$: Therefore $2AC \times BD + BD^2 = BC^2 - AC^2$; and, by adding AC^2, common, $AC^2 + 2AC \times BD + BD^2$ (or the ſquare of AC + BD, *by* 6. 2.) will be $= BC^2$; and, conſequently, AC + BD = BC. *Q. E. D.*

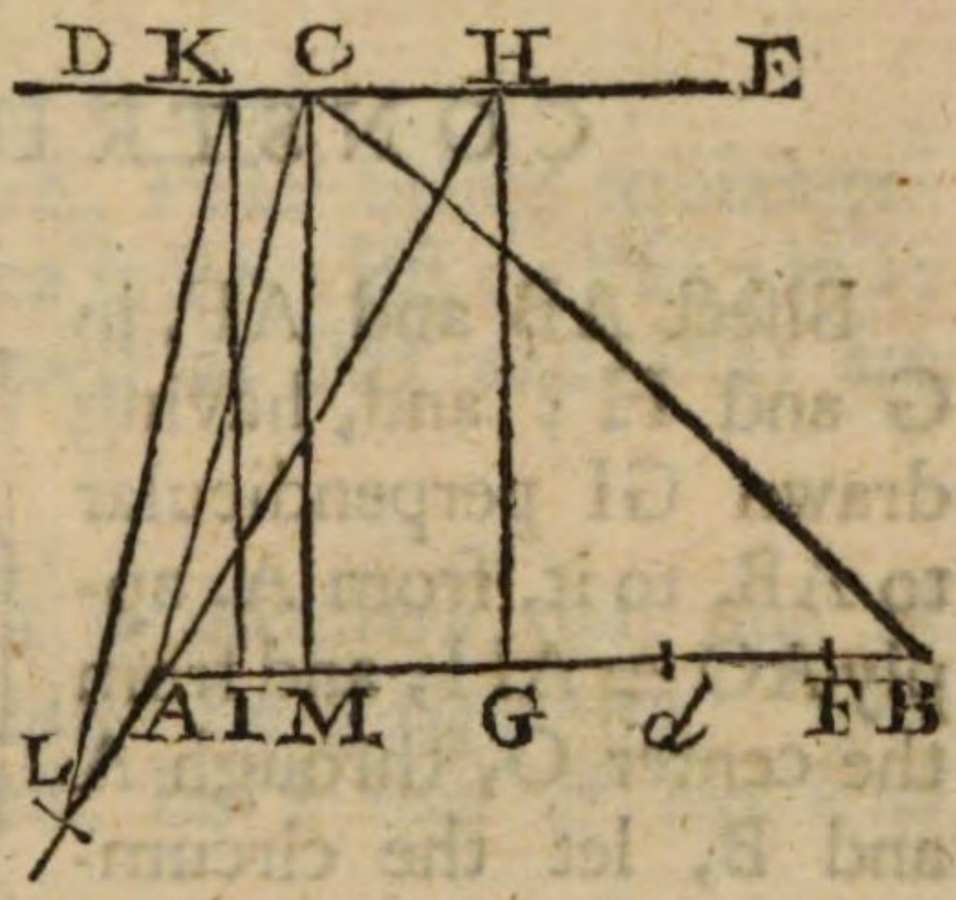
This *Lemma* is not only of uſe in the Problem next following, but will be found a ready inſtrument in the ſolution of many others; for which reaſon it is here put down.

PRO-

PROBLEM XV.

From two given points A, B, *to draw two lines* AC, BC *to meet in a right-line* DE, *given by position; so that their difference shall be equal to a given right-line* Bd.

CONSTRUCTION.



In AB, take a third-proportional BF to BA and B*d*; and, having bisected AF in G, take GI = B*d*; make GH and IK perpendi- to AB, meeting DE in H and K; and draw HAL, to which, from K, apply KL = AB; and parallel thereto draw AC, meeting DE in C; join B, C; *and the thing is done.*

DEMONSTRATION.

Let CM be perpendicular to AB.

Then, becauſe of the parallel lines, it will be AC : KL (AB) : : HC : HK : : GM : GI (B*d*). Whence *(by alternation)* AC being to GM, as BA to B*d*; and AB, *d*B, FB being alſo proportionals; the whole Conſtruction is manifeſt *from the Lemma* premiſed.

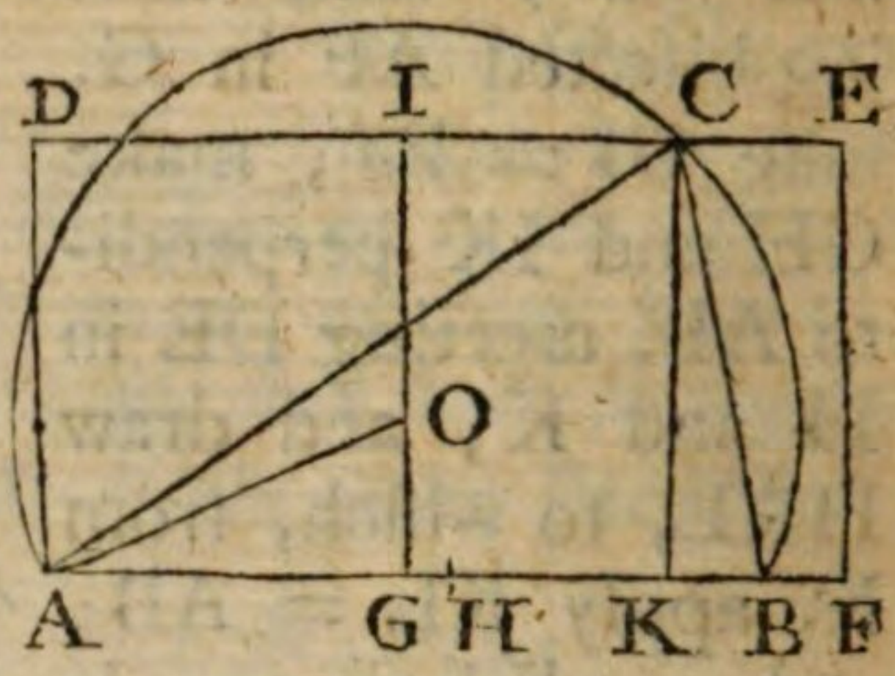
If the ſum, inſtead of the difference, of the two lines (AC, BC) be given, the method of conſtruction will be exactly the ſame, without the leaſt alteration of any one ſtep; provided that B*d* be firſt of all taken (in BA, produced) equal to the given ſum, inſtead of the difference.

PRO-

PROBLEM XVI.

From two given points A, B, *to draw two lines* AC, BC, *so as to meet in a right-line* DE *parallel to that* (AB) *joining the said points, and that the rectangle* (AC × BC) *contained by them, shall be equal to a rectangle given,* ADEF.

CONSTRUCTION.



Bisect AB and AF, in G and H; and, having drawn GI perpendicular to AB, to it, from A, apply AO = AH; and from the center O, through A and B, let the circumference of a circle be described intersecting DE in C; from whence draw CA and CB, *and the thing is done.*

For, if CK be drawn perpendicular to AB, it is evident *(by* 25. 3.) that AC × BC = CK × 2AO = AD × AF. *Q. E. D.*

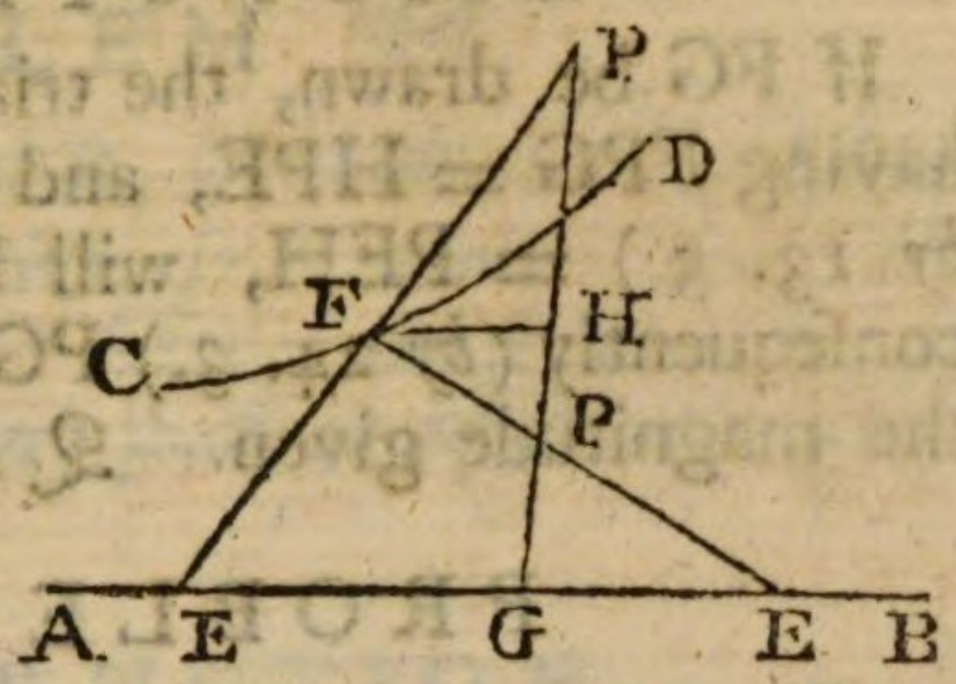
PROBLEM XVII.

Through a given point P, *so to draw a line* FPE, *that the parts thereof* PF, PE, *intercepted by that point and two lines* AB, CD, *given in position, shall obtain a given ratio.*

CON-

CONSTRUCTION.

Through P, to any point in AB, draw PG, in which (*by* 13. 5.) take PH to PG, in the given ratio of PF to PE; draw HF parallel to AB, meeting CD in F; then draw FPE, *and the thing is done.*



For, the triangles PGE, PHF being equiangular (*by* 3. *and* 7. 2.) thence is PF : PE : : PH : PG: And ſo PF and PE (as well as PH and PG) are in the ratio given.

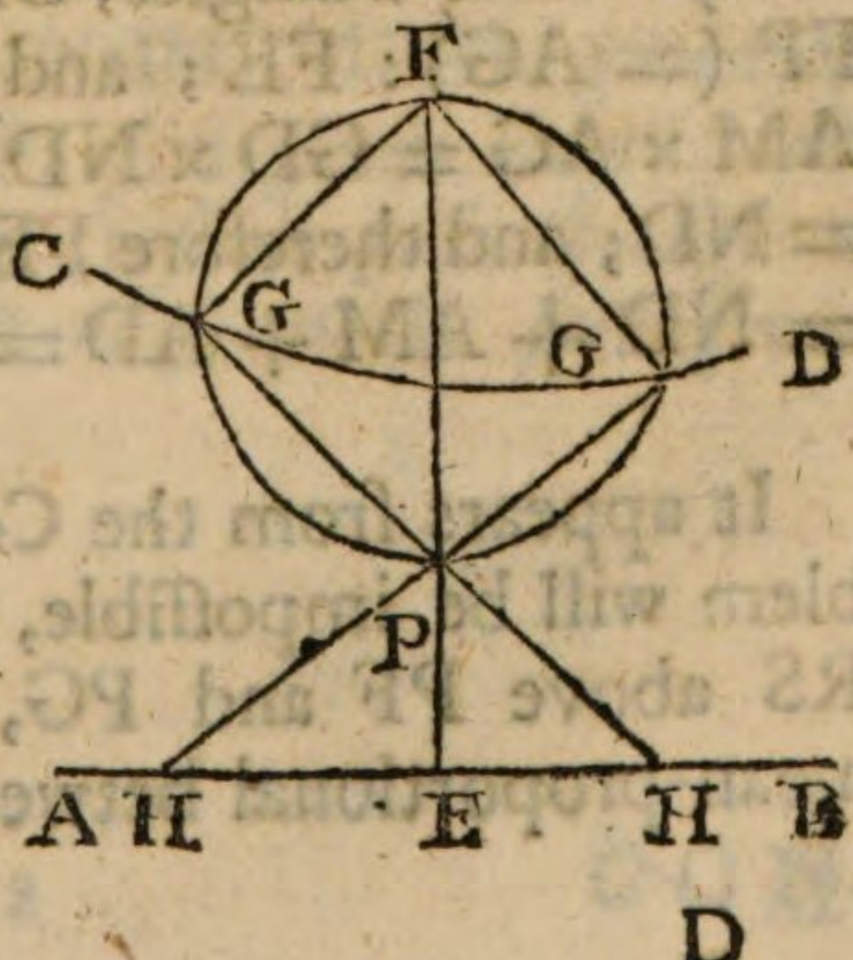
In this Conſtruction, it is neceſſary that one of the two given lines ſhould be a *right-one*: The other may, it is manifeſt, be a line of *any kind* whatever.

PROBLEM XVIII.

Through a given point P, *to draw a line* GH, *terminating in a right-line* AB, *and in a line* CD *of any kind, given both by poſition; ſo that the rectangle under the two parts thereof* PG, PH, *ſhall be of a given magnitude.*

CONSTRUCTION.

Having drawn EPF perpendicular to AB, take therein PF (*by* 7. 6.) ſo that PE × PF = the magnitude given: Upon PF, as a diameter, let a circle be deſcribed, interſecting CD in G; then draw GPH, *and the thing is done.*



DEMONSTRATION.

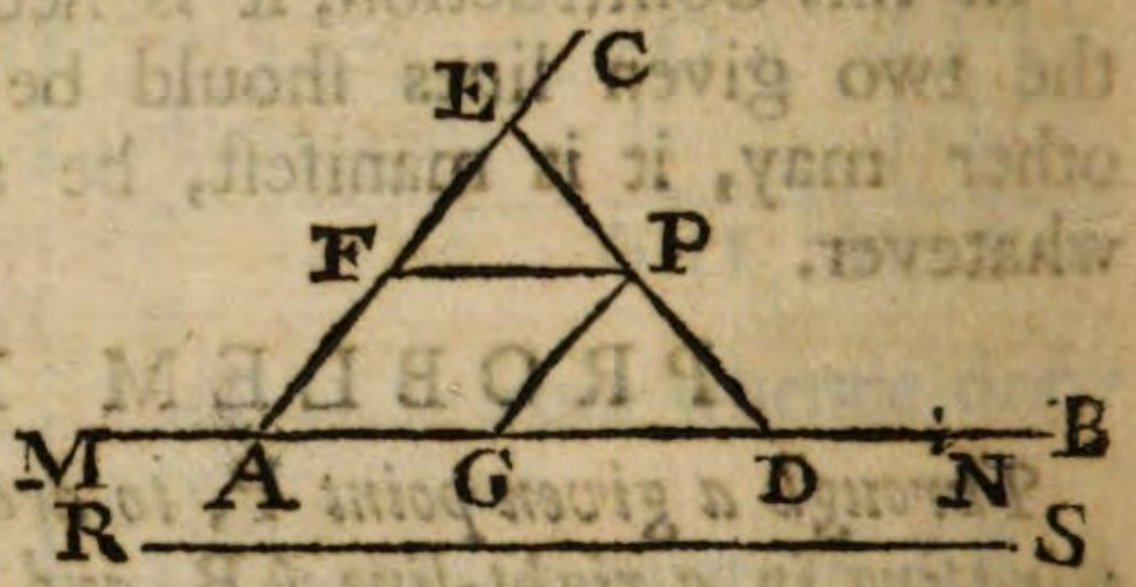
If FG be drawn, the triangles PFG and PHE, having FPG = HPE, and FGP (= a right-angle, *by* 13. 3.) = PEH, will be equiangular: And, consequently (*by* 24. 3.) PG × PH = PF × PE = the magnitude given. *Q. E. D.*

PROBLEM XIX.

Through a given point P, *between two right-lines* AB, AC, *given by position, so to draw a line* ED, *that the sum of the segments* (AD + AE) *cut off by it from the two former, shall be equal to a given line* RS.

CONSTRUCTION.

Draw PF and PG parallel to AB and AC; in BA produced take AM = AF, and MN = RS; then divide GN in D (*by* 17. 5.) so that GD × ND = AM × AG; draw DPE, *and the thing is done.*



DEMONSTRATION.

By similar triangles, GD : GP (= AF = AM) :: FP (= AG) : FE; and consequently GD × FE = AM × AG = GD × ND (*by Constr.*): Whence FE = ND; and therefore FE + AF + AD (AE + AD) = ND + AM + AD = MN = RS. *Q. E. D.*

It appears from the Conſtruction, that the Problem will be impoſſible, when (GN) the exceſs of RS above PF and PG, exceeds the double of a mean-proportional between theſe two quantities.

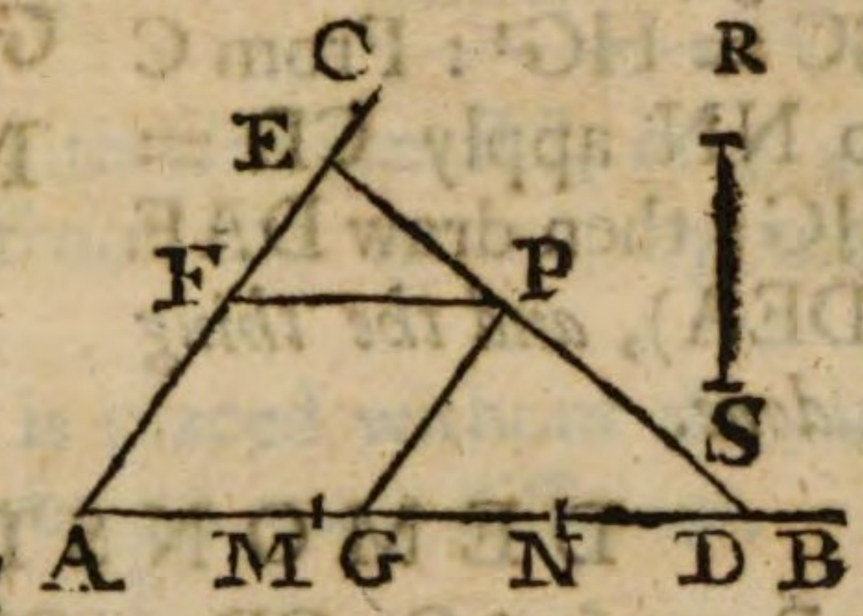
PRO-

PROBLEM XX.

Through a given point P, *between two right-lines* AB, AC, *given by position, ſo to draw a line* DE, *that the difference* (AD — AE) *of the ſegments cut off by it from the two former, ſhall be equal to a given line* RS.

CONSTRUCTION.

Draw PF and PG parallel to AB and AC; in AB take AM = AF, and MN = RS; then (*by* 18. 5.) let a line ND be added to GN, ſo that GD × ND, PG × PF; draw DPE, *and the thing is done.*



DEMONSTRATION.

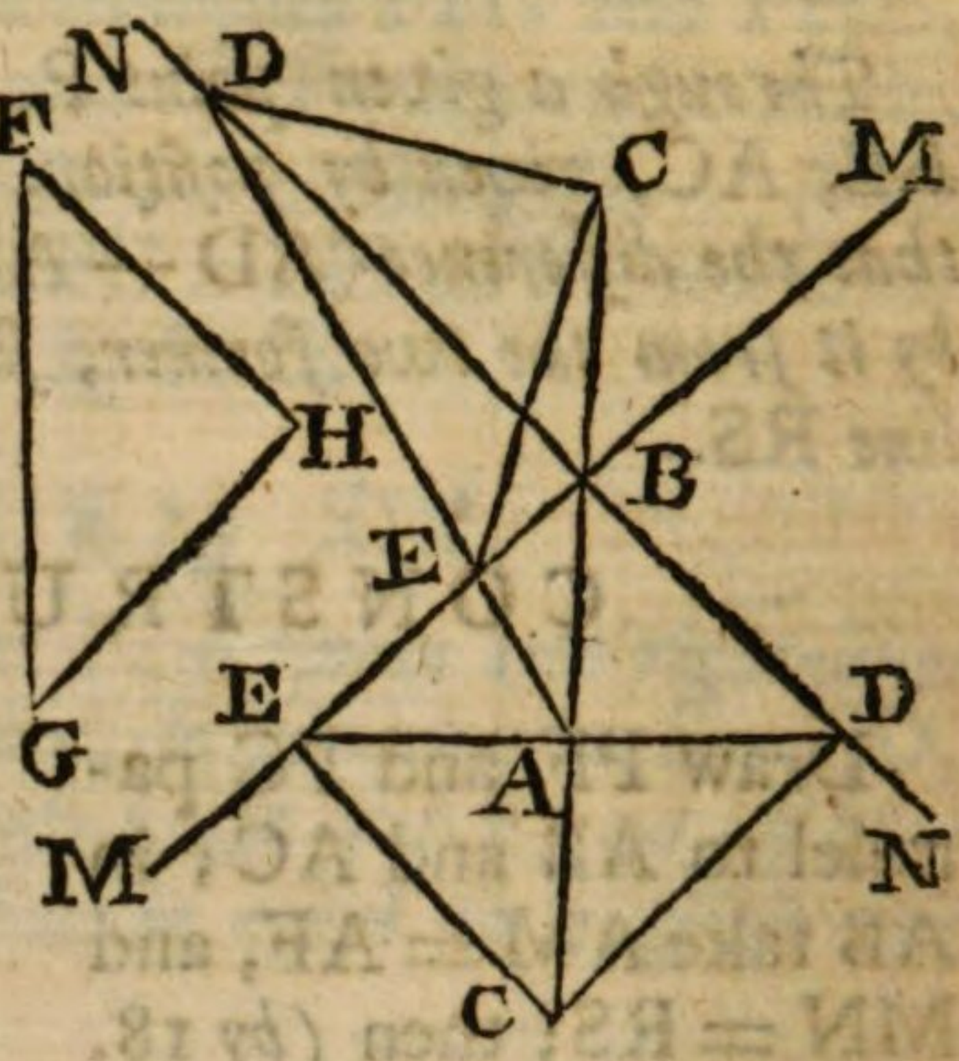
Becauſe of the ſimilar triangles PGD, EFP, we have GD × FE = PG × PF = GD × ND (*by Conſtr.*) and conſequently FE = ND; whence AD — AE = AN — AF (AM) = MN = RS. *Q.E.D.*

PROBLEM XXI.

Between two right-lines NBN, MBM, *given by poſition, to apply a line* DE *of a given length, and which (produced, if neceſſary) ſhall paſs through a given point* A, *equally diſtant from the ſaid given lines.*

CONSTRUCTION.

Let FG be the length given; and, having drawn AB, make the angles F, G equal each to ABD (or ABE); and in AB produced take AC *(by* 18. 5.) ſo that AC × BC = HG²: From C to NN apply CD = HG; then draw DAE (DEA), *and the thing is done.*



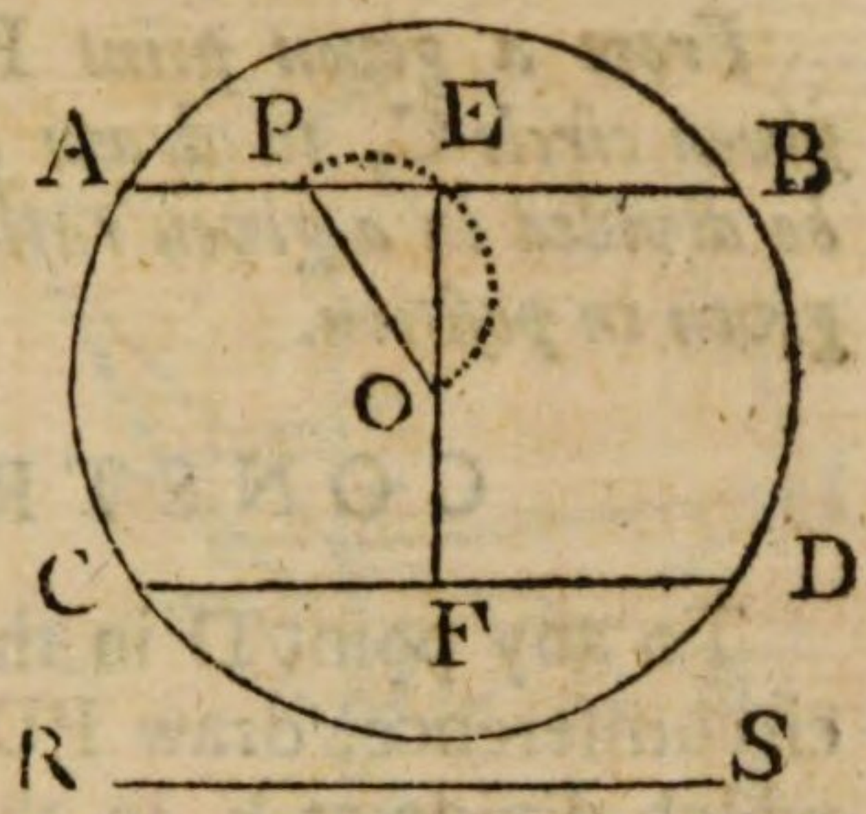
DEMONSTRATION.

Becauſe AC : CD : : CD : BC *(by Conſtr. and* 10. 4.) the triangles CAD and CDB (having one angle common, and the ſides about it proportional) will be equiangular (*by* 15. 4.): And therefore, ſince CDA = DBC = ABE *(by hypotheſis)*, the circumference of a circle may be deſcribed through all the four points C, D, B, E (*by* 11. 3. *or* 19. 3.) And ſo the angle DEC, ſtanding on the ſame ſubtenſe (DC) with DBC, will be equal to it; and, conſequently, equal to EDC. Therefore the iſoſceles triangles EDC, FGH being equiangular, and having CD = HG, their baſes ED and FG will alſo be equal. *Q. E. D.*

PROBLEM XXII.

In a given circle ABDC, *to apply a chord* AB *equal to a given line* RS *(leſs than the diameter) and which ſhall paſs through a given point* P.

CONSTRUCTION.



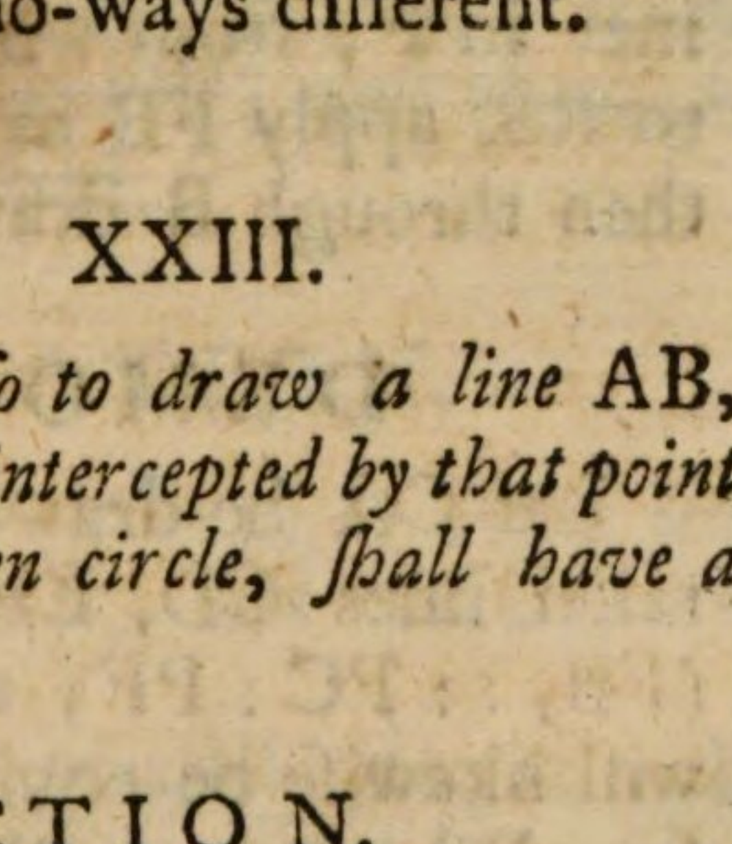
Inſcribe CD = RS, upon which, from the center O, let fall the perpendicular OF; alſo draw OP, upon which let a ſemi-circle be deſcribed, and in it apply OE = OF; laſtly, through P and E, let AB be drawn; which will, manifeſtly, be equal to CD (= RS, *by* 3. 3.) as being equally diſtant from the center, by conſtruction.

When the point given is placed without the circle, the conſtruction will be no-ways different.

PROBLEM XXIII.

Through a given point P, *ſo to draw a line* AB, *that the parts thereof* AP, BP, *intercepted by that point and the circumference of a given circle, ſhall have a given difference,* DE.

CONSTRUCTION.

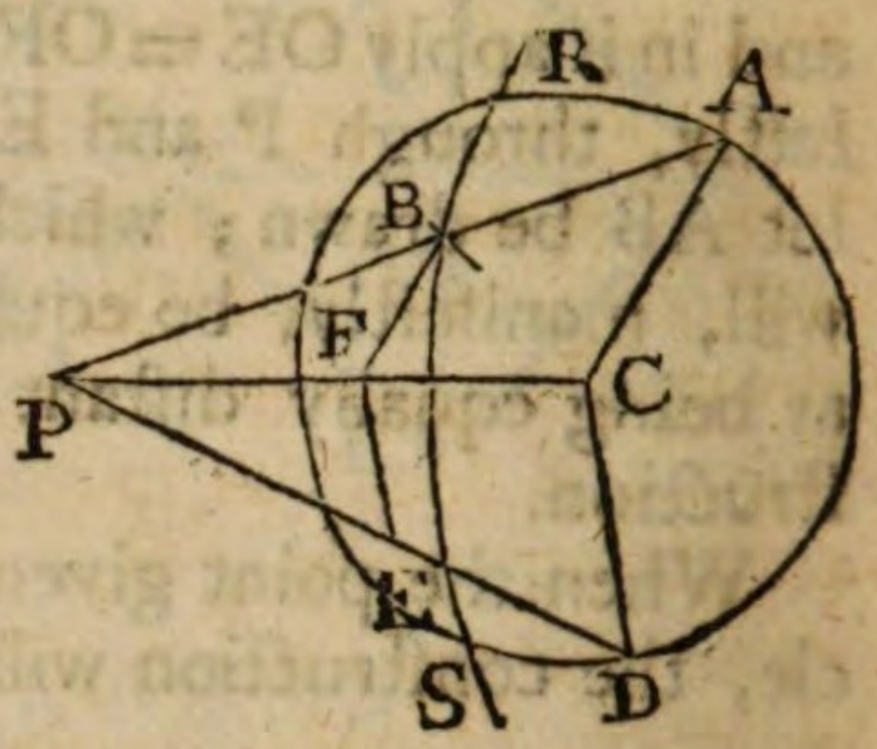


From the center C, draw CP, upon which let a ſemi-circle PQC be deſcribed, and in it apply PQ equal to half DE, producing the ſame, both ways, to meet the circumference in A and B: So ſhall AQ = BQ (*by* 2. 3); and therefore PB (= BQ + PQ = AQ + PQ = AP + 2PQ) = AP + DE, *which was to be done.*

PROBLEM XXIV.

From a given point P, *to the circumference of a given circle* C, *to draw a right-line* PBA, *ſo as to be divided in a given ratio by a line* RBS *of any kind, given in poſition.*

CONSTRUCTION.



To any point D in the circumference, draw PD, which divide at E in the ratio given; and, having drawn PC and the radius CD, parallel to the latter draw EF, meeting the former in F; from whence, to RS, apply FB = FE;
then through B draw PA, *and the thing is done.*

DEMONSTRATION.

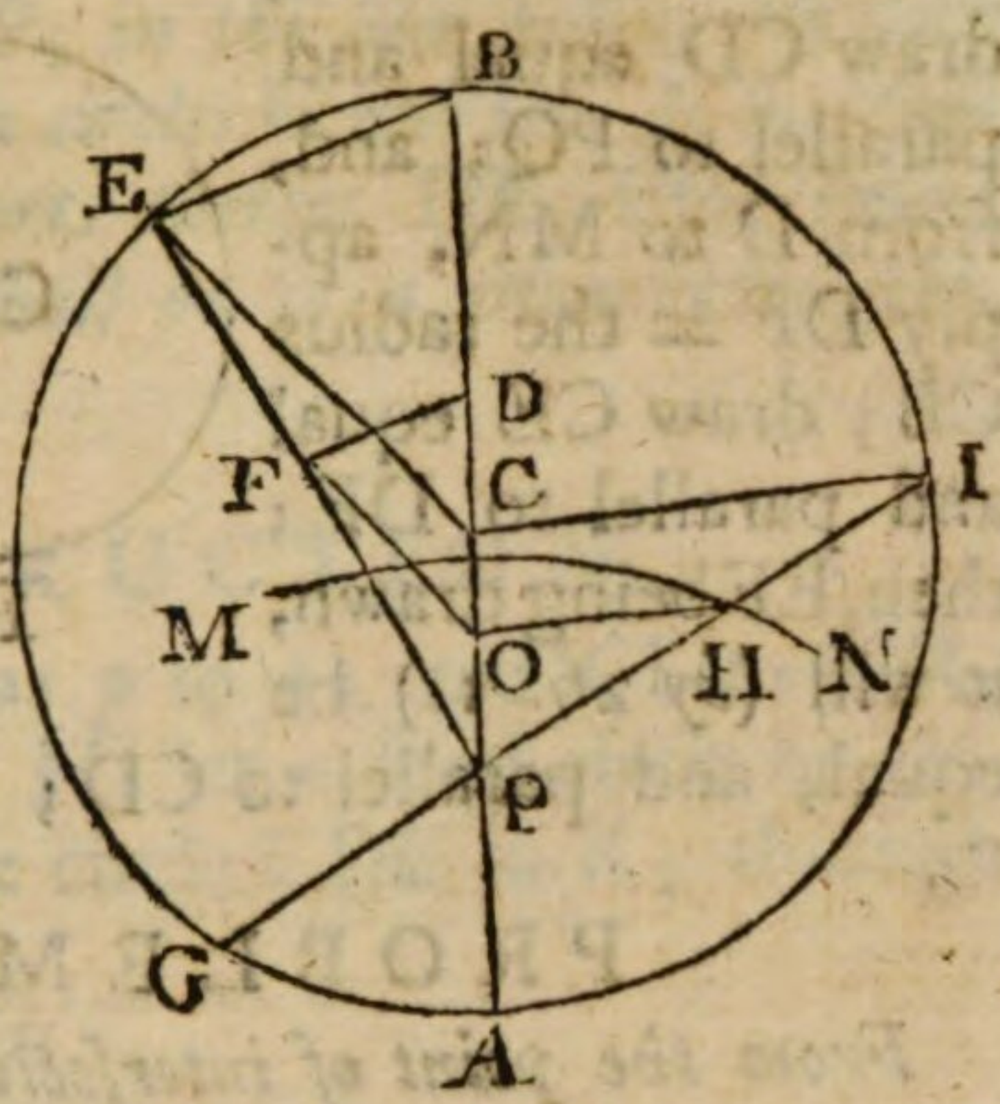
Let CA be drawn. Then, becauſe of the parallel lines CD, EF, it will be, CD (AC) : EF (FB) : : PC : PF; and ſo the triangles PAC, PBF will likewiſe be equiangular (*by* 16. 4.) And therefore PB : BA : : PF : FC : : PE : ED (*by* 13. 4.) *Q. E. D.*

PROBLEM XXV.

Through a given point P, *to draw a line* GH, *terminating in the circumference of a given circle* BGA, *and in a line* MN *of any kind, given by poſition; ſo that the rectangle under the two parts thereof* PG, PH, *ſhall be of a given magnitude.*

CON-

CONSTRUCTION.



Through P draw the diameter AB, in which let PD be so taken (*by* 7. 6.) that PA × PD = the magnitude given. From any point E in the circumference, draw EP and the radius EC; and, having joined BE, draw DF parallel to BE, and from its interfection with PE, draw FO parallel to EC, meeting PB in O; from whence, to MN, apply OH = OF; and through P draw HG for the line required.

DEMONSTRATION.

Let PH be produced (if neceffary) to meet the circumference of the circle in I; and let C, I be joined.

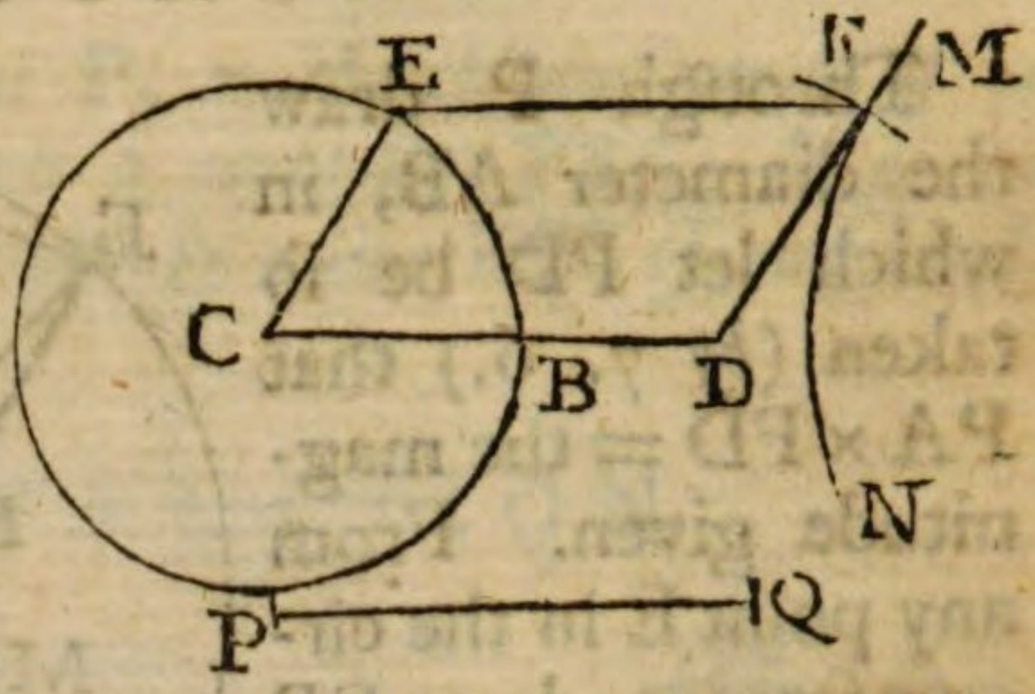
The lines OF, CE being parallel, thence will PO : PC : : OF (OH) : CE (CI); and therefore OH and CI will likewife be parallels (*by* 16. 4.) Therefore PH : PI : : PO : PC : : PF : PE : : PD : PB; whence, alternately, PH : PD : : PI : PB : : PA : PG (*by* 21. 3. *and* 10. 4); and confequently PH × PG = PD × PA. *Q. E. D.*

PROBLEM XXVI.

From the circumference of a given circle C, *to a line* MN *of any kind, given by pofition, fo to draw a right-line* EF, *as to be both equal and parallel to a given right-line* PQ.

CONSTRUCTION.

From the center C, draw CD equal and parallel to PQ; and, from D to MN, apply DF = the radius CB; draw CE equal and parallel to DF; then, EF being drawn, it will (*by* 26. 1.) be equal, and parallel to CD; *which was to be done.*

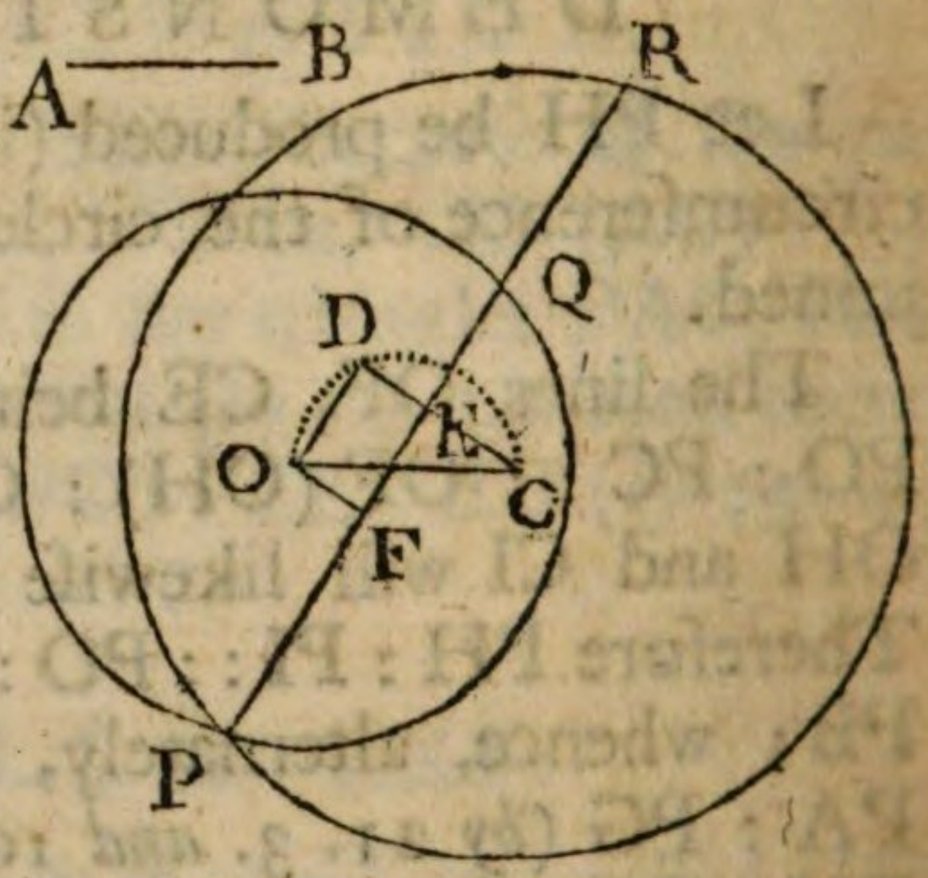


PROBLEM XXVII.

From the point of interſection P *of two given circles* O *and* C, *ſo to draw a line* PR, *that the part thereof* QR *intercepted by the two peripheries, ſhall be equal to a given line* AB.

CONSTRUCTION.

Upon the line OC joining the two centers, let a ſemi-circle ODC be deſcribed, in which apply OD = $\frac{1}{2}$AB; and parallel thereto, draw PR, *and the thing is done.*



DEMONSTRATION.

Let CD, and OF parallel to CD, be drawn, meeting PR in E and F. Then, the angle ODC being a right-one (*by* 13. 3.) and PR parallel to DO, E and F will alſo be right-angles, and EF = DO (*by* 24. 1): And ſo, PE — PF being (= DO) = $\frac{1}{2}$AB, it is manifeſt, that 2PE (PR) — 2PF (PQ) = AB, or that QR = AB. *Q. E. D.*

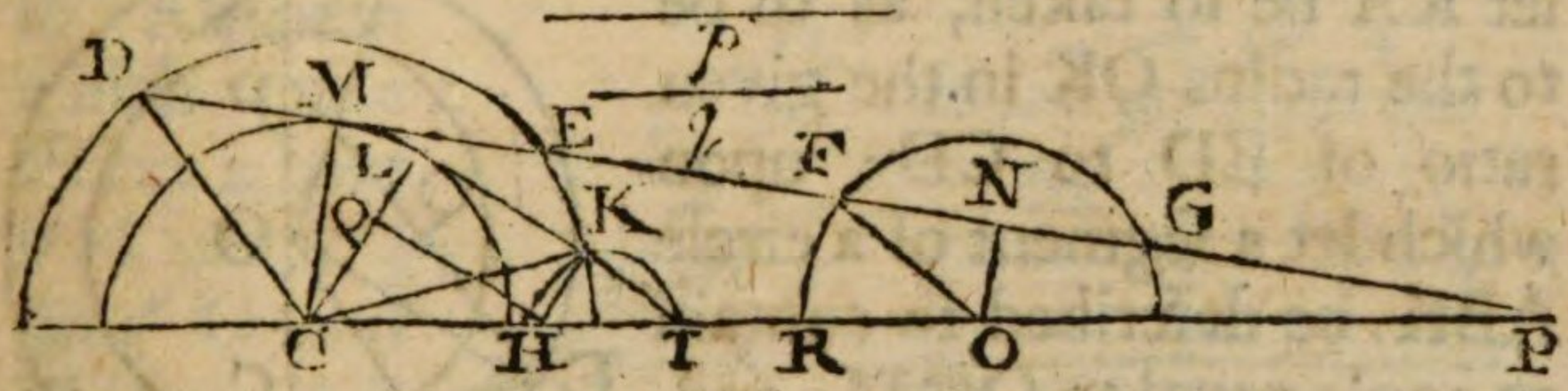
PRO-

PROBLEM XXVIII.

From a given point P, *in the line paſſing through the centers of two given circles* C *and* O; *ſo to draw a line* PD, *that the parts thereof* DE, FG, *intercepted by thoſe circles, ſhall be in a given ratio* (viz. *as* p *is to* q).

CONSTRUCTION.

Take CH to OR, as *p* to *q*; and CI to OR, as PC to PO: Upon HI let a ſemi-circle be deſcribed, interſecting the circle CDE in K; through



which point draw IKL, and make CL perpendicular thereto; at which diſtance, from the center C, let a circle ML be deſcribed: Then, if from P a line PD be drawn to touch this circle, *the thing is done.*

DEMONSTRATION.

Let CD, CK, HK, FO be drawn, and alſo CM and ON, perpendicular to PD, and HQ to CL.

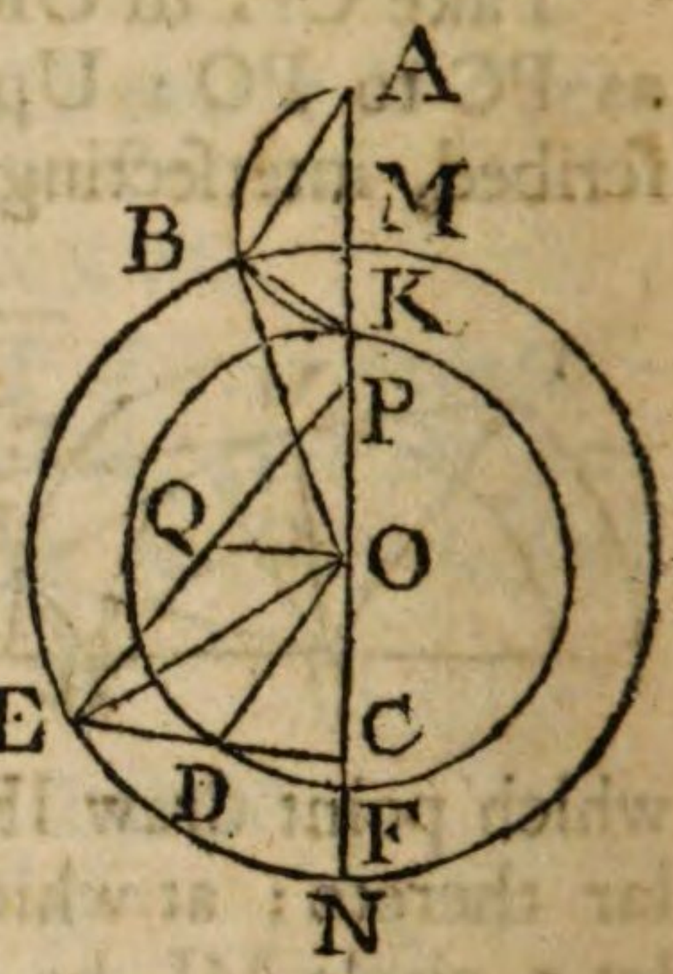
The right-angled triangles CDM, CKL, having CD = CK, and CM = CL, have alſo DM = LK = QH (ſince, *by* 13. 3. the angle HKI, as well as L, is a right-one). Moreover *(by Conſtruction)* CI : OR (OF) : : PC : PO : : CM (CL) : ON; and ſo the triangles CIL, OFN (having their ſides proportional) muſt be ſimilar; and conſequently OFN alſo ſimilar to CHQ: whence, as QH (or DM) : FN : : CH : FO (OR) : : *p* : *q (by Conſtr.)* *Q. E. D.*

PRO-

PROBLEM XXIX.

To draw a line EC, *to make given angles with a line* MN *passing through the common center* O *of two given circles* MEN, KDF, *so that the parts thereof* CD, ED, *intercepted by that line and the two peripheries, shall obtain a given ratio.*

CONSTRUCTION.



Let QOM be the given angle, to which ECM shall be equal: In OM produced, let KA be so taken, as to be to the radius OK in the given ratio of ED to CD; upon which let a segment of a circle ABK be described to contain an angle equal to QOM; and, from its intersection with the circle MEN, draw BA; parallel to which, draw the radius OD; and then, through D, draw EC parallel to QO.

DEMONSTRATION.

Let BK, BO, EO be drawn, and also EP parallel to DO, meeting OQ and OM in Q and P.

Then, the triangles AKB, POQ, having ABK = POQ (*by Construction*), and KAB = OPQ (*by* 7. 1.) have their external angles OKB, EQO also equal (*by* 9. 1); Therefore, because EQ (= OD) = OK, and EO = BO, thence is OQ = KB (*by* 17. 1); and therefore, as the triangles POQ, ABK are equiangular, PQ is also = AK: But (because of the parallel lines) CD : ED :: DO (OK) : PQ (AK). *Q. E. D.*

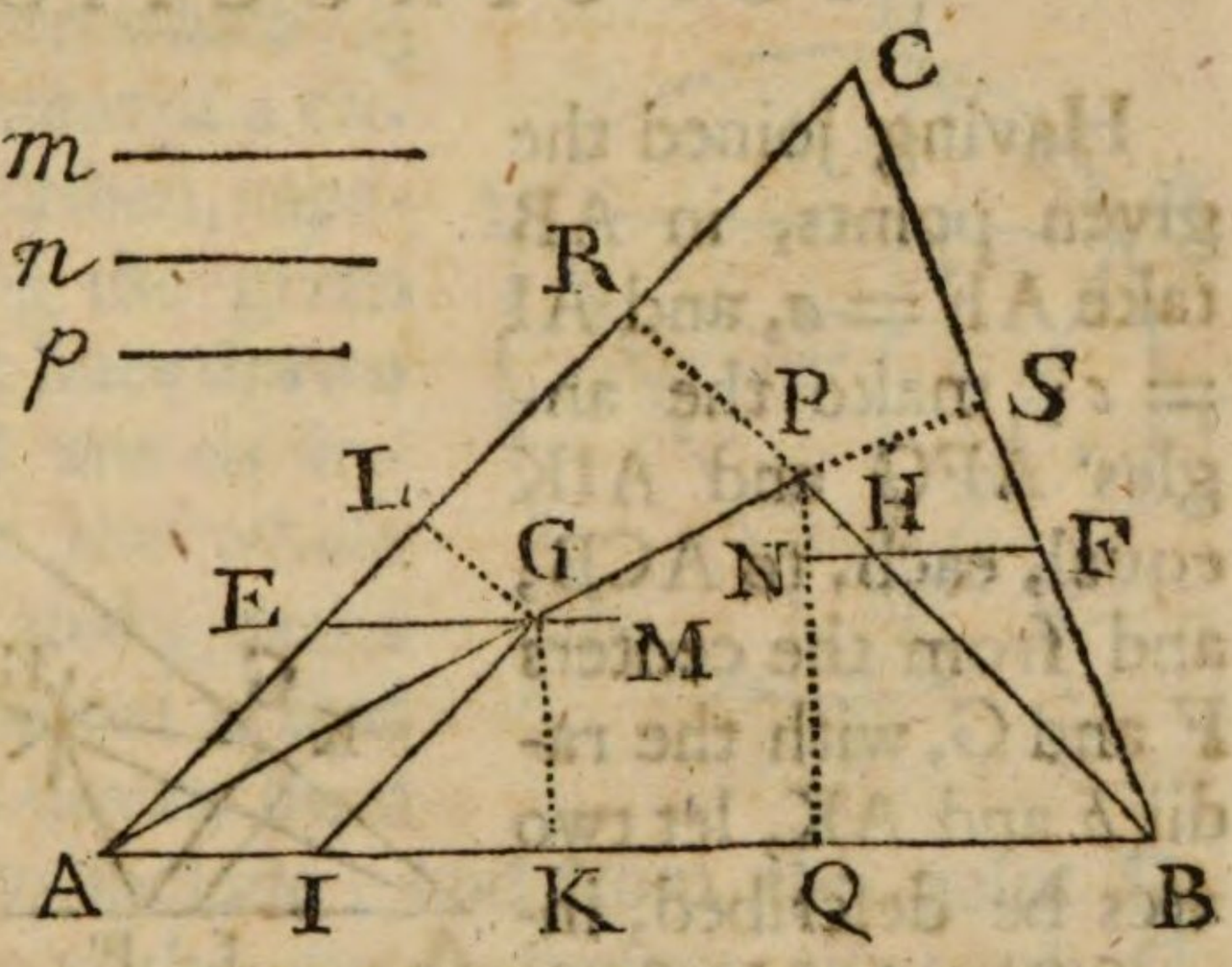
PRO-

PROBLEM XXX.

To determine a point P, *ſo that three perpendiculars drawn from thence, to as many right-lines* AB, AC, BC, *given by poſition, ſhall obtain the ratio of three given lines* m, n *and* p, *reſpectively.*

CONSTRUCTION.

Take AE and BF each $= m$, and draw EM and FN parallel to AB; in which take $EG = n$, and $FH = p$; thro' G and H draw AP and BP, and the point of concourſe P, *will be that which was to be determined.*



DEMONSTRATION.

Upon the ſides of the triangle ABC let fall the perpendiculars GL, GK, PR, PQ, PS; and let GI, parallel to AC, be drawn.

The angle LEG being $=$ LAK $=$ GIK (*Cor.* 1. *to* 7. 1.) and L and K being both right-angles (*by Conſtr.*), the triangles EGL and GIK are ſimilar; and therefore IG (m) : EG (n) : : GK : GL : : PQ : PR (*by* 21. 4). And, in the very ſame manner, $m : p$: : PQ : PS. *Q. E. D.*

After the ſame way, the Problem may be conſtructed, when the lines drawn from P are required to make any given angles with the lines upon which they fall.

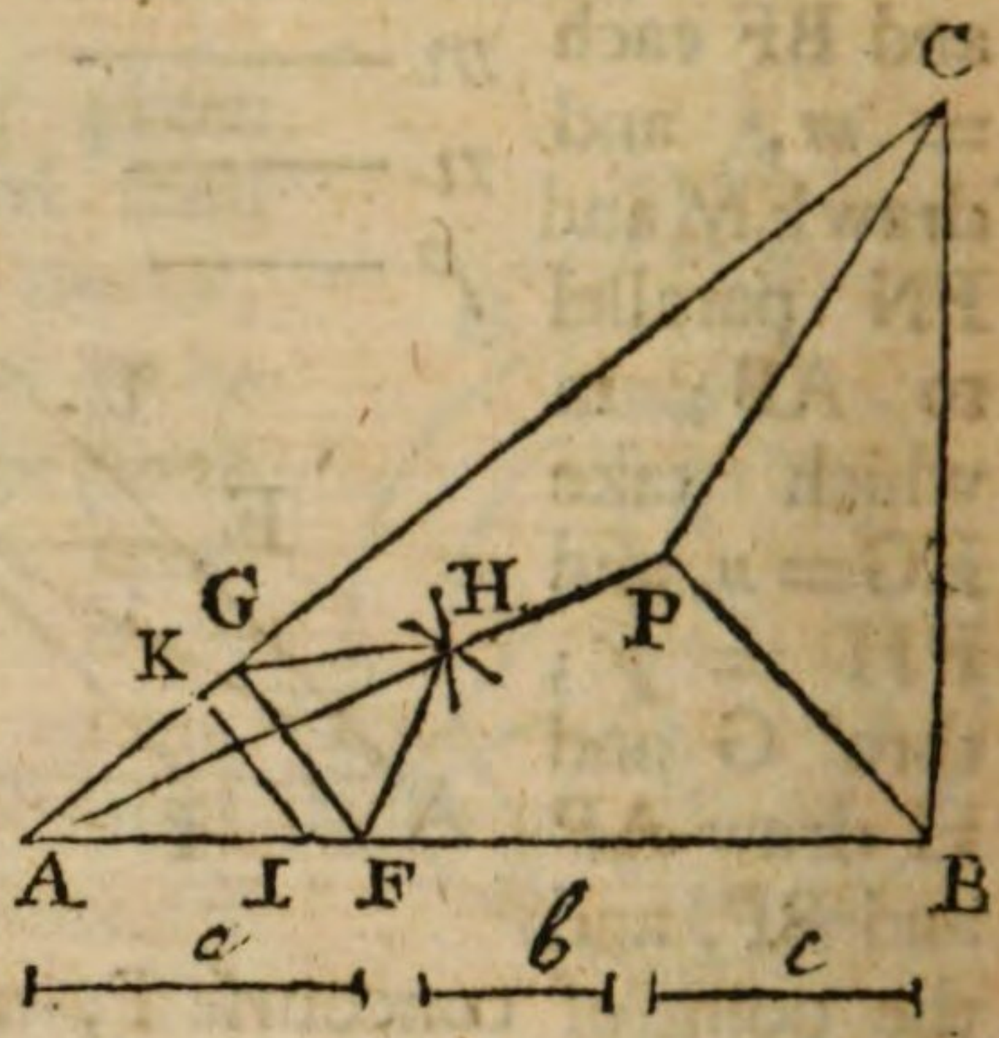
PRO-

PROBLEM XXXI.

To determine a point P, *ſo that three lines* PA, PB, PC, *drawn from thence to three given points* A, B, C, *ſhall obtain the ratio of three given lines* a, b *and* c, *reſpectively.*

CONSTRUCTION.

Having joined the given points, in AB take AF = *a*, and AI = *c*; make the angles AFG and AIK equal, each, to ACB; and from the centers F and G, with the radii *b* and AK, let two arcs be deſcribed, interſecting in H; from which point draw HF and HA; then draw BP to make the angle ABP = AHF, and it will meet AH (produced) in the point P, required.



DEMONSTRATION.

Let BP, CP, and GH be drawn.

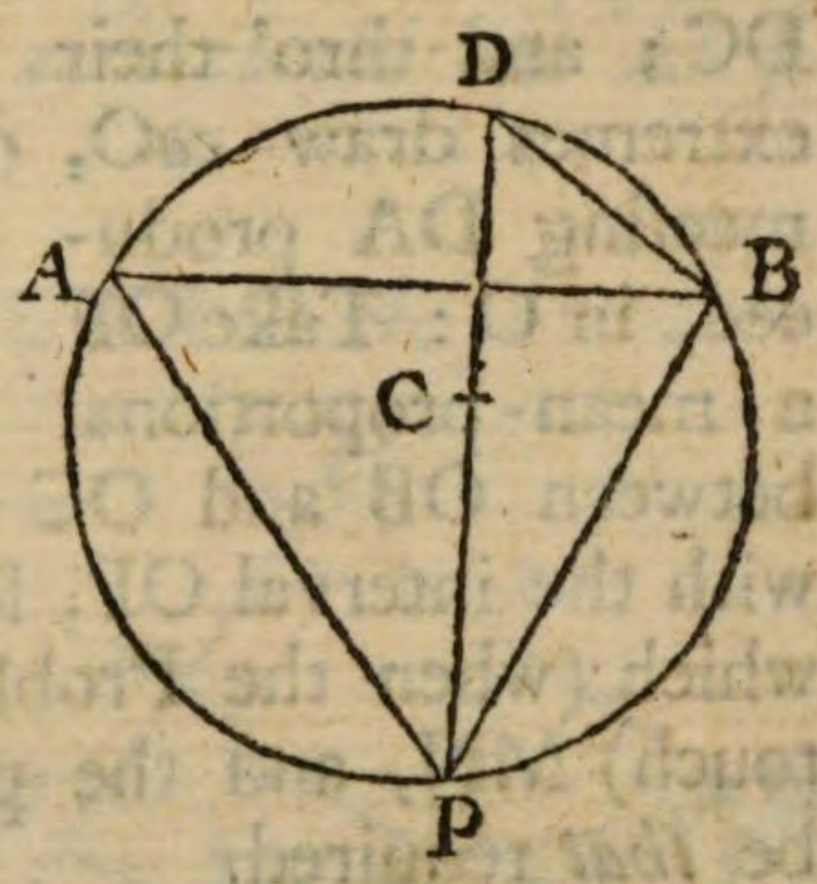
The triangles ABP, AHF being equiangular, it will be, AP : BP : : AF (*a*) : FH (*b*); alſo AB : AP : : AH : AF; and AB : AC : : AG : AF (becauſe ABC and AGF are likewiſe equiangular, *(by Conſtr.)*. Now, ſeeing the extremes of the two laſt proportions are the ſame, the four means AP, AC, AG, AH *(by* 10. 4.) will therefore be proportionals; and ſo, the triangles ACP, AHG being equiangular *(by* 15. 4.) it will be AP : CP : : AG : GH (AK) : : FA (*a*) : AI (*c*). *Q. E. D.*

PRO-

PROBLEM XXXII.

To determine the position of a point P, *at which, lines drawn from three given points* A, B, C, *shall make given angles, one with another.*

CONSTRUCTION.



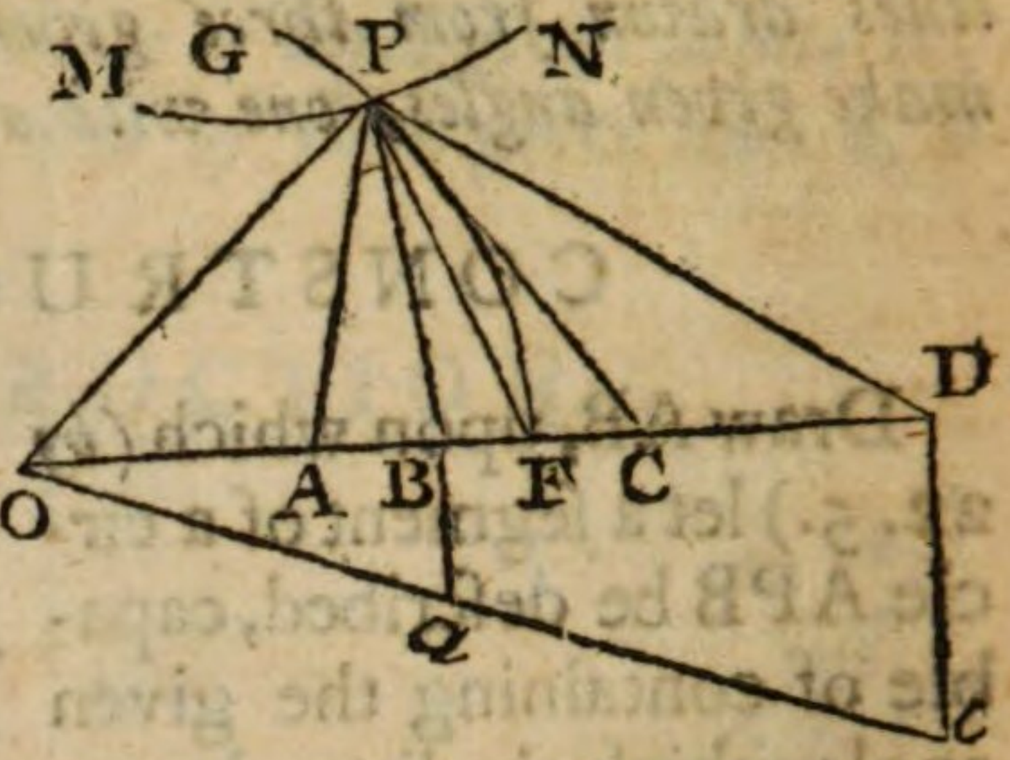
Draw AB, upon which (*by* 22. 5.) let a segment of a circle APB be described, capable of containing the given angle which the lines drawn from A and B are to include; and let the whole circle be completed; make the angle ABD equal to that which the lines drawn from A and C are to include; and from the point D, where BD meets the circumference, through C, let DP be drawn, meeting the circumference in P; *which is the point required.*

For, AP and BP being drawn, the angle APC will be = the given angle ABD (*by* 11. 3.) both standing on the same arch AD: And APB is also of the given magnitude by construction.

PROBLEM XXXIII.

Any two unequal segments AB, CD *of a right-line* AD *being given, as well in position as length; to determine a point* P *in a line* MN *of any kind, given by position; at which the two angles* APB, CPD, *subtended by those segments, shall (if possible) be equal to each other.*

CONSTRUCTION.



Make B*a* and D*c* parallel to each other, taking the former = BA, and the latter = DC; and thro' their extremes draw *ca*O, meeting DA produced, in O: Take OF a mean-proportional between OB and OC; and from the center O, with the interval OF, let a circle FG be described; which (when the Problem is possible) will cut (or touch) MN, and the point of intersection P, will be *that* required.

DEMONSTRATION.

Let PO, PA, PB, PF, PC, and PD be drawn.

Because OD : D*c* (DC) : : OB : B*a* (BA), it will be *(by division)* OD : OC : : OB : OA; and consequently $OD \times OA = OC \times OB = OP^2$ *(by Constr.)*. Therefore, seeing that OA : OP : : OP : OD, and that the angle O is common to both the triangles OAP, OPD, these triangles must be equiangular (*by* 15. 4.) and consequently APF = OPF (OFP) = OPA (ODP) = DPF (*by* 9. 1). In the very same manner, because OB : OP : : OP : OC, the angle BPF will be = CPF; and, consequently, APB also = CPD. *Q. E. D.*

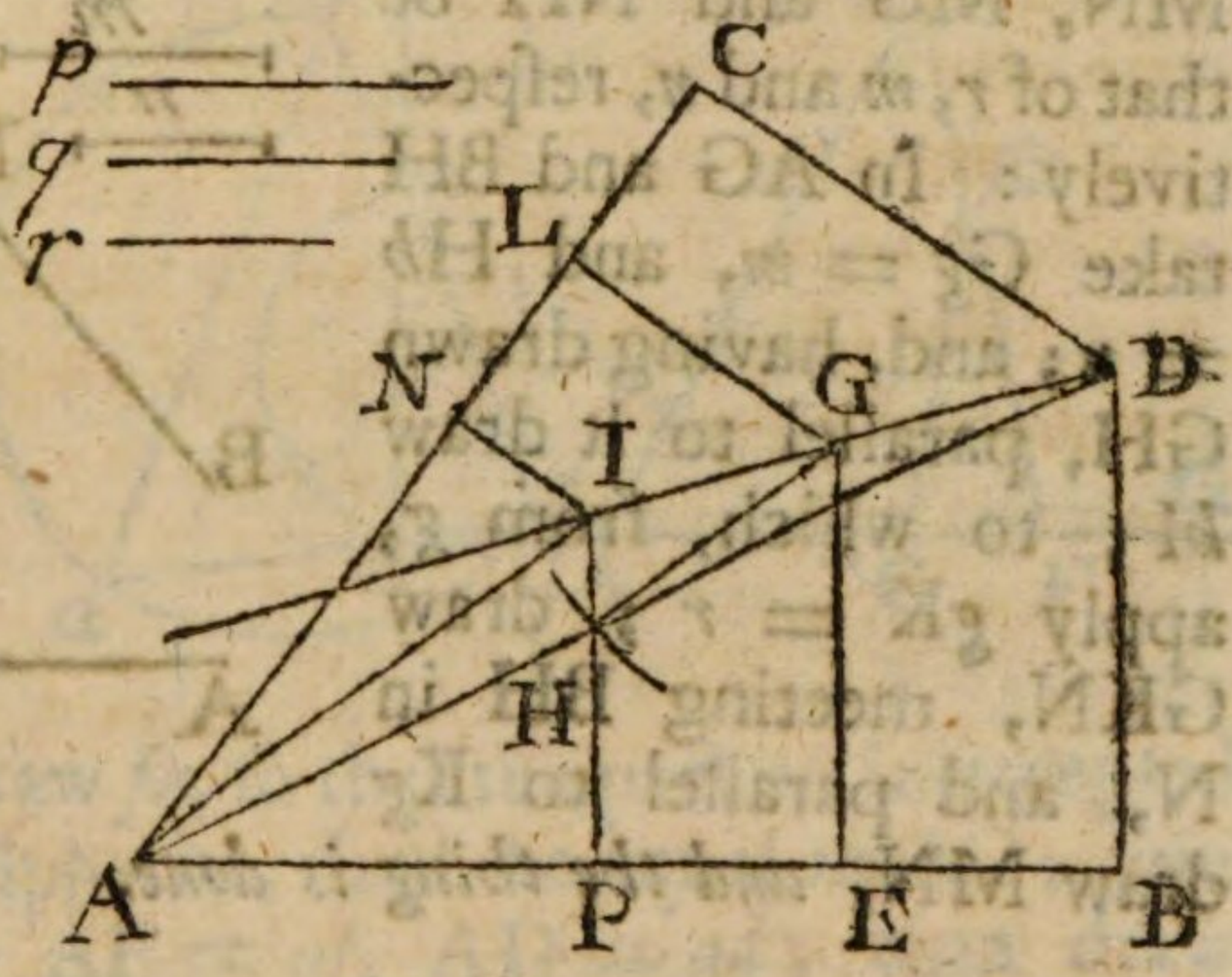
PRO-

PROBLEM XXXIV.

Two right-lines AB, AC *being given, both in length and position; from the point of their concourse* A, *so to draw another line* AI, *that two perpendiculars* IP, IN, *falling from the extreme thereof upon the two given lines, shall cut off alternate segments* BP, CN *in a given ratio each to the line* AI *so drawn.*

CONSTRUCTION.

Let the given ratio of AI, BP and CN be that of the lines *p*, *q* and *r*, respectively. Take BE $= q$, and CL $= r$; making BD, EG, CD and LG perpendicular to AB and AC, so as to meet in D and G: Draw DA and DG, and from G to AD, apply GH $= p$, and parallel thereto draw AI, meeting DG produced (if needful) in I, *and the thing is done.*



DEMONSTRATION.

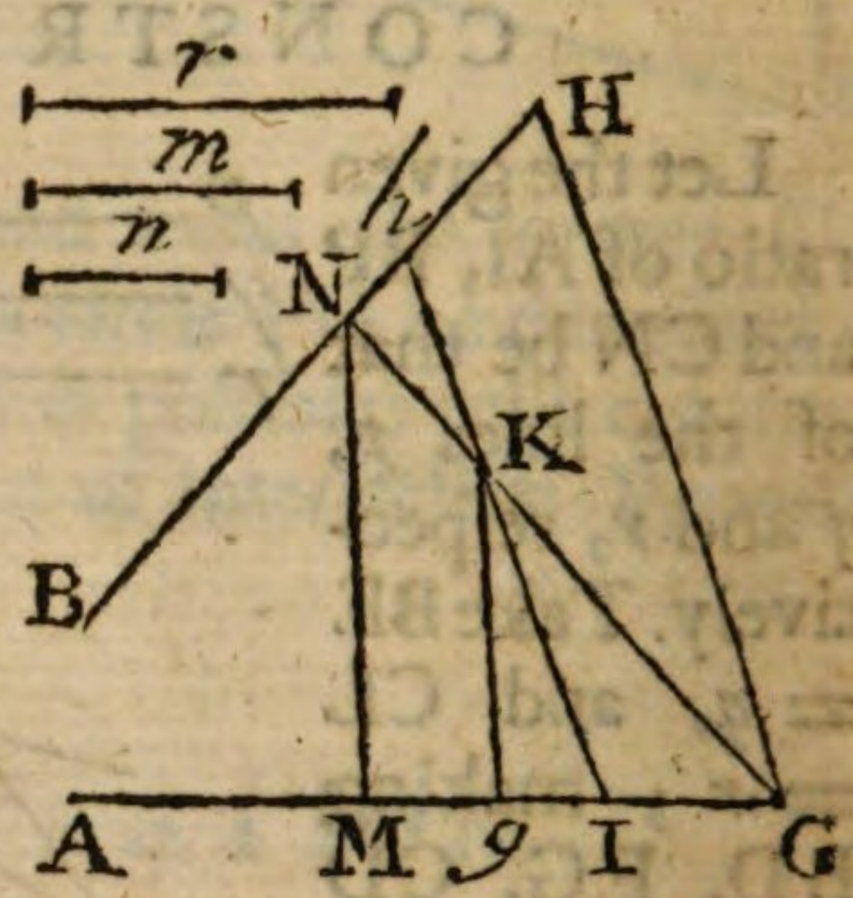
Becauſe of the parallel lines, AI : GH (: : ID : GD) : : BP : BE; whence, alternately, AI : BP : : GH (p) : BE (q). In the very ſame manner AI : CN : : p : r. *Q. E. D.*

PRO-

PROBLEM XXXV.

Between two lines AG, BH, *given both in position and length, to draw a line* MN, *which shall be in a given ratio to each of the segments* MG, NH *cut off from the two given lines.*

CONSTRUCTION.



Let the given ratio of MN, MG and NH be that of r, m and n, respectively: In AG and BH take $Gg = m$, and $Hb = n$; and, having drawn GH, parallel to it draw bI; to which, from g, apply $gK = r$; draw GKN, meeting BH in N, and parallel to Kg draw MN, *and the thing is done.*

DEMONSTRATION.

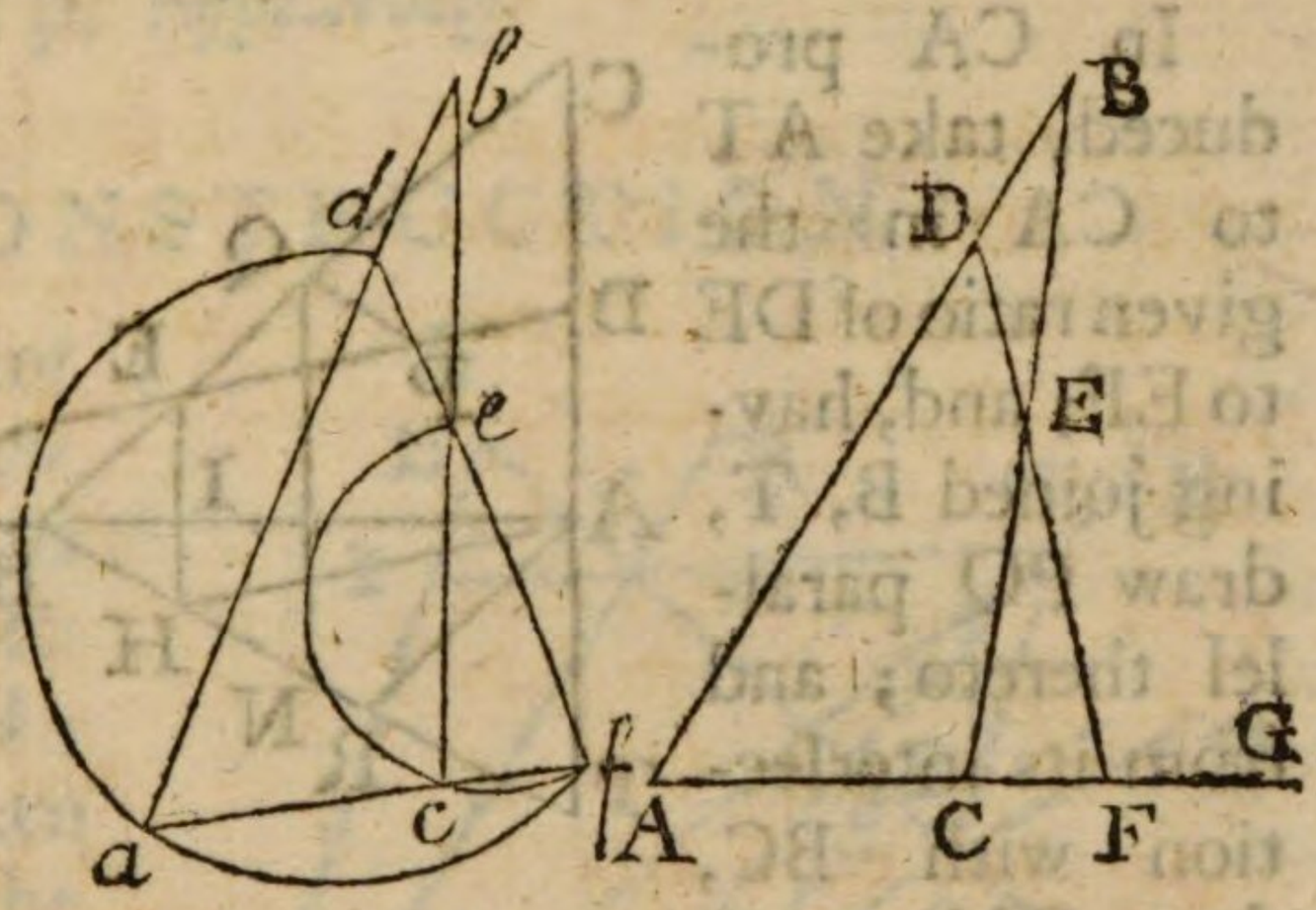
Becauſe of the parallel lines, it will therefore be MN : GM :: gK (r) : Gg (m); and likewiſe MN : gK (r) :: GN : GK :: NH : Hb (n); which laſt *(by alternation)* is MN : NH :: r : n. *Q. E. D.*

PRO-

PROBLEM XXXVI.

To draw a line DF, *to cut three other lines* AB, AG, BC, *given by poſition; ſo that the two parts thereof* DE, EF, *intercepted by thoſe lines, ſhall be reſpectively equal to two given lines,* de, ef.

CONSTRUCTION.



Upon the right-lines *df*, *ef* let two ſegments of circles, *daf*, *ecf*, be deſcribed, to contain angles, reſpectively equal to BAG and BCG: Then (*by Prob.*27.) draw *fa*, ſo that the part *ac*, intercepted by the two peripheries, ſhall be equal to AC; join *a*, *d*, and take AF = *af*, AD = *ad*; then draw DF, *and the thing is done.*

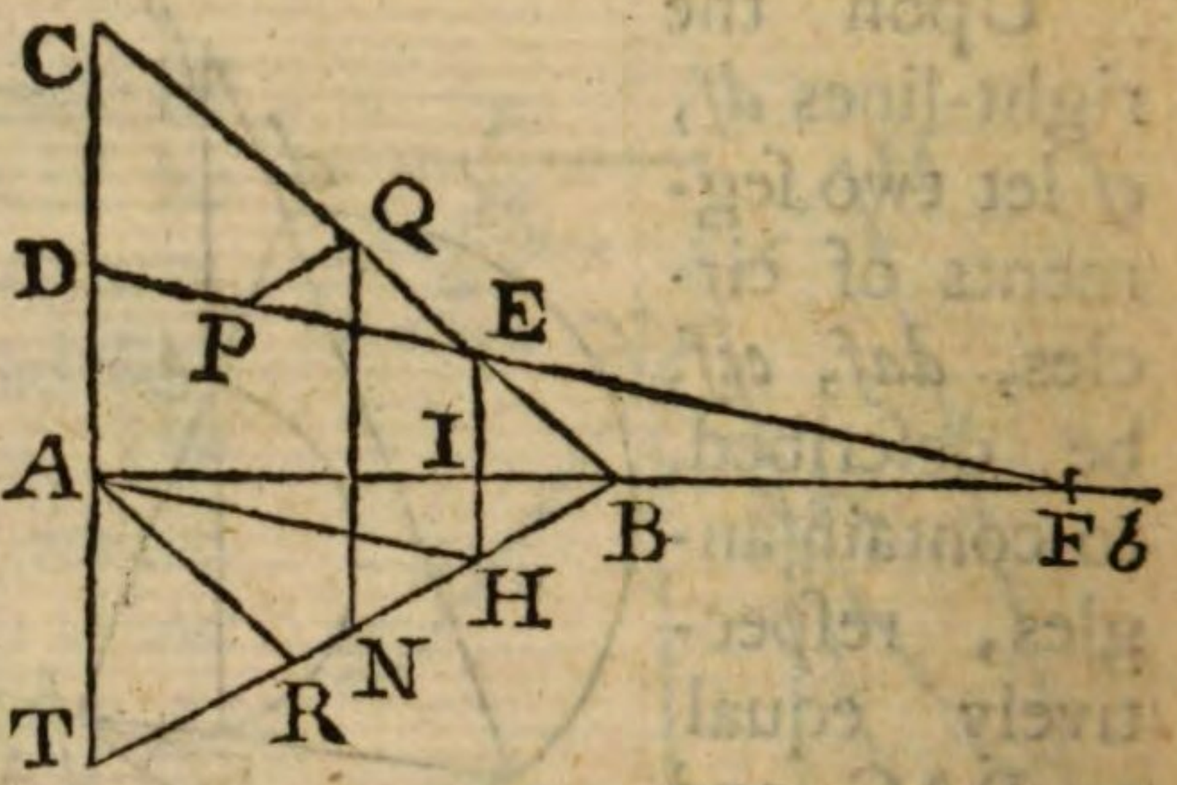
DEMONSTRATION.

The triangles FAD, *fad*, having AF = *af*, AD = *ad*, and A = *a* (*by Conſtruction*), are equal in all reſpects; and therefore (if *ce* be drawn) the triangles FCE, *fce*, having F = *f*, FCE = *fce*, and CF = *cf*, will alſo be equal and alike: Therefore, ſeeing the wholes DF, *df*, and the parts FE, *fe* are equal, the remaining parts DE and *de* muſt likewiſe be equal. *Q. E. D.*

PROBLEM XXXVII.

Through a given point P, *to draw a line* DPF, *to cut three lines* AC, CB, ABb, *given by position; ſo that the parts thereof* DE, EF *intercepted by thoſe lines, ſhall obtain a given ratio.*

CONSTRUCTION.



In CA produced, take AT to CA in the given ratio of DE to EF; and, having joined B, T, draw PQ parallel thereto; and from its interſection with BC, draw QN parallel to CT; alſo draw AR parallel to BC: And in RB take RH *(by* 18. 5.) ſo that NH × RH = PQ × TR: Then draw AH, and DPEF parallel to it, *and the thing is done.*

DEMONSTRATION.

Becauſe *(by Conſtr. and* 10. 4.) NH : TR :: PQ : RH :: QE : AR *(by* 14. 4.) it follows *(by alternation)* that NH : QE :: TR : AR :: TB : BC *(by* 12. 4.) :: BN : BQ: Therefore EH (when drawn) will be parallel to QN or AD *(by Cor. to* 12. 4); and ſo DEHA being a parallelogram, we have again (by ſimilar triangles) as DE (AH) : EF :: IH : IE :: AT : AC. *Q. E. D.*

When the ſegments AD, BE, cut off from the given lines AC, BC, are required to be in a given ratio (inſtead of DE and EF), the conſtruction will be

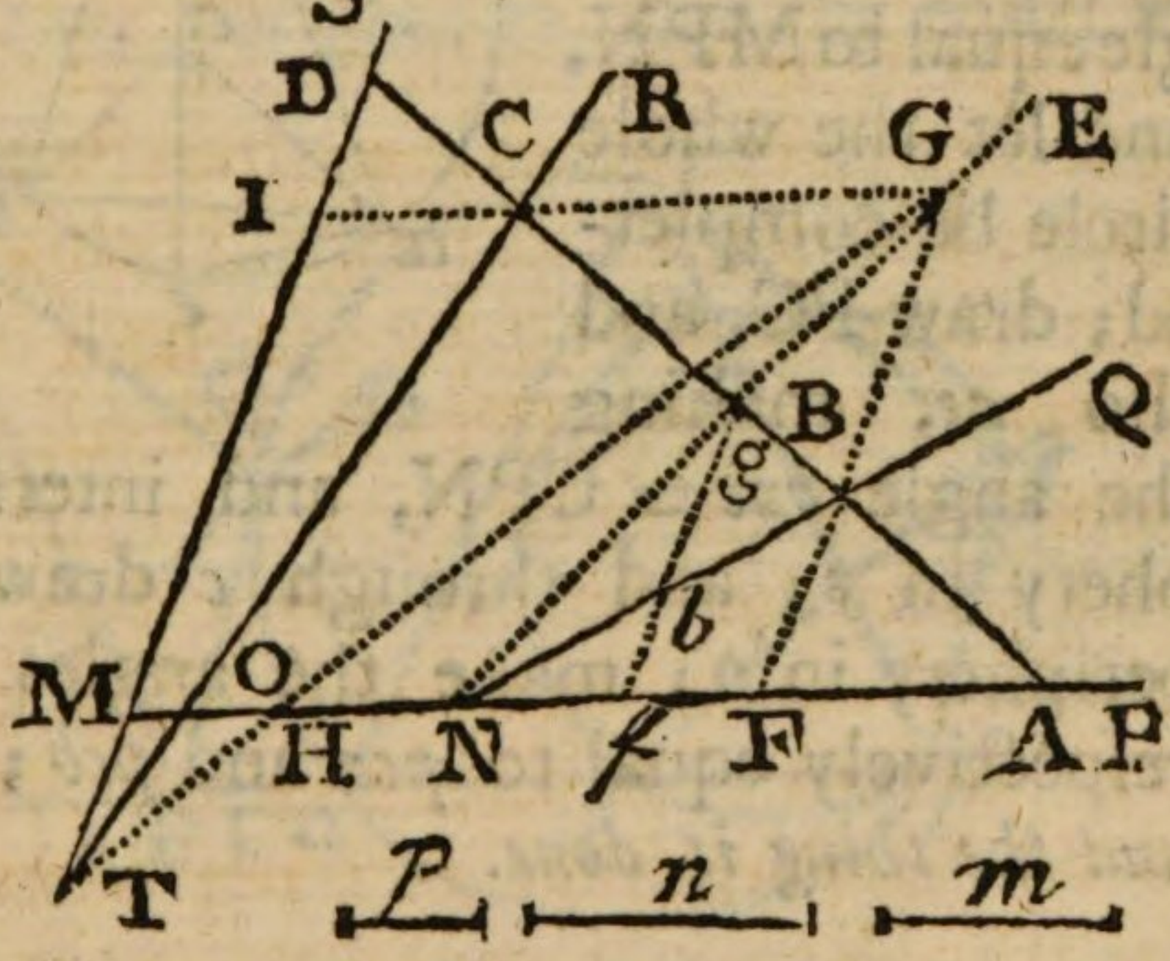
be the ſame; provided that CT be taken to BC in the ratio given. For, then AD (EH) : BE : : CT : BC.

PROBLEM XXXVIII.

To draw a line ABCD *to cut four other lines* MONP, NQ, ROT, SMT, *given by poſition; ſo that the three parts thereof* AB, BC, CD, *intercepted by thoſe lines, may obtain the ratio of three given lines* m, n, *and* p, *reſpectively.*

CONSTRUCTION.

From any point *f* in NP draw *fbg* parallel to TMS, interſecting NQ in *b*; and, having taken *bg* to *bf* in the given ratio of *n* to *m*, draw NgE: Moreover, having taken OH to MO in the given ratio of *n* to *p*, draw THG; and from (G) its interſection with NE, draw GI and GF parallel to MP and ST, reſpectively, cutting NQ and TR in B and C; thro' which two points draw ABCD, *and the thing is done.*



DEMONSTRATION.

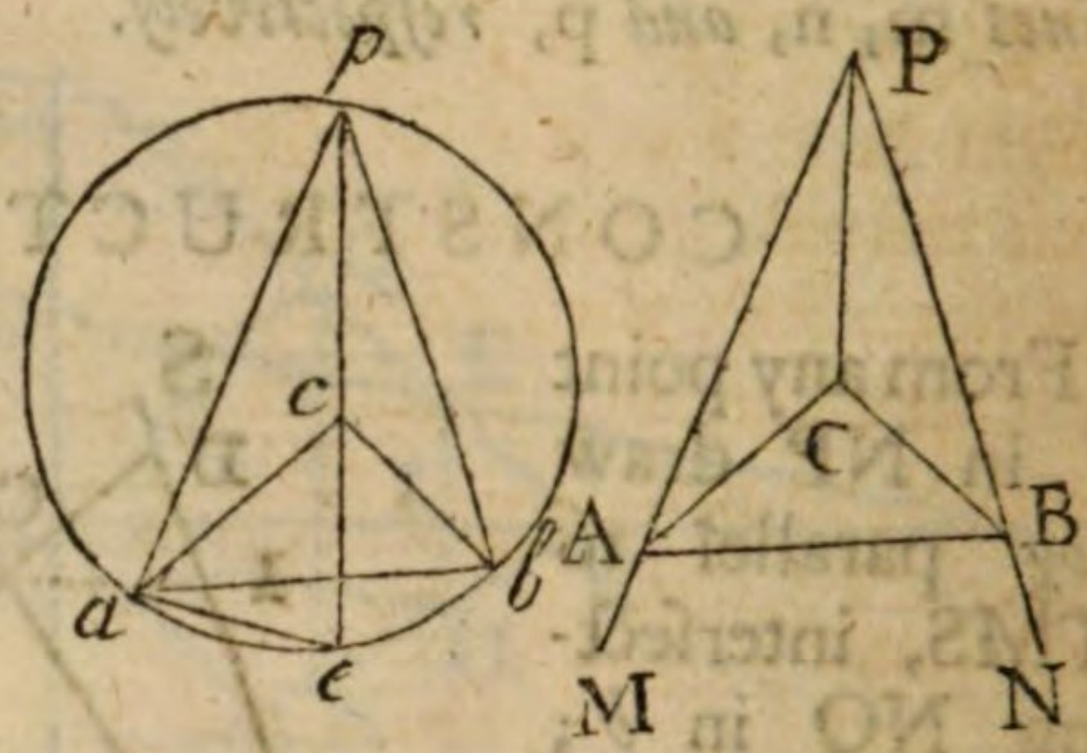
By ſimilar triangles { AB : BC : : BF : BG : : *bf* : *bg* : : *m* : *n*; BC : CD : : CG : CI : : OH : MO : : *n* : *p*.

Q. E. D.

PROBLEM XXXIX.

To draw two lines CA, CB, *from a given point* C, *to terminate in two other lines* PM, PN, *given by position, and which, together with the line* AB, *joining their extremes, ſhall form a triangle,* ABC, *ſimilar to a given one,* abc.

CONSTRUCTION.



Upon *ab* let a ſegment of a circle *apb* be deſcribed, to contain an angle equal to MPN, and let the whole circle be completed; draw PC, and alſo *ae*, making the angle *bae* = CPN, and interſecting the periphery in *e*; and through *c* draw *ep*, meeting the periphery in *p*; make the angles PCA and PCB reſpectively equal to *pca* and *pcb*; then join A, B, *and the thing is done.*

DEMONSTRATION.

If *pa* and *pb* be drawn; then will the angle *bae* (CPB) = *cpb*, both ſtanding on the ſame arch *be*; therefore, APB being alſo = *apb* (*by Conſtr.*), the remaining angles APC and *apc* muſt conſequently be equal; whence, as PCA = *pca*, and PCB = *pcb* (*by Conſtr.*), the triangles APC, *apc*, and BPC, *bpc* are equiangular; and therefore AC : *ac* (: : PC : *pc*) : : CB : *cb*. And ſo the triangles ABC, *abc*, having the ſides about the equal angles ACB, *acb*, proportional, they are like to each other (*by* 15. 4). *Q. E. D.*

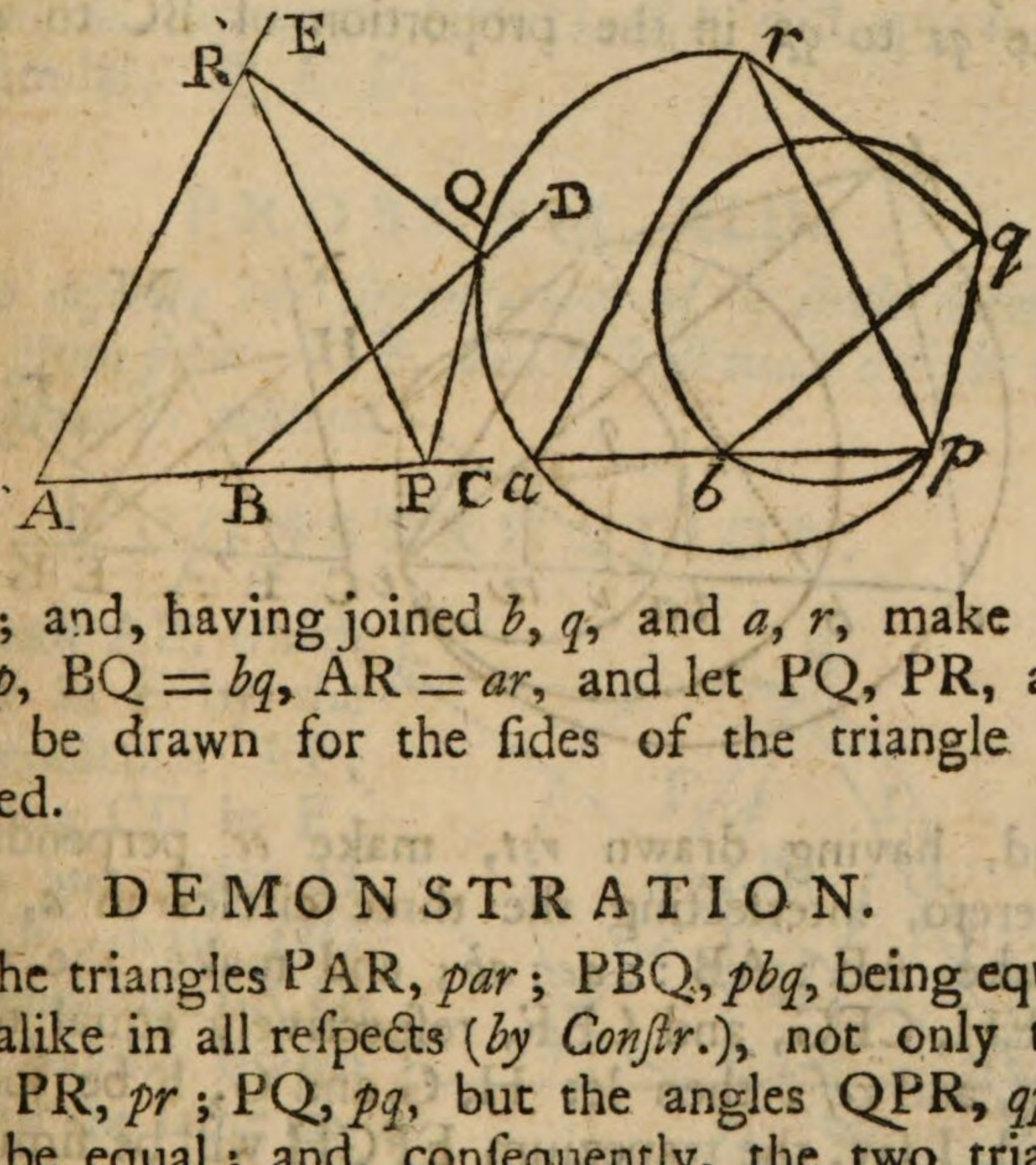
PRO-

PROBLEM XL.

To describe a triangle PQR, *equal and similar to a given triangle* pqr, *which shall have its angular points placed in three right-lines* ABC, BD, AE, *given by position.*

CONSTRUCTION.

Upon the two sides *pq*, *pr*, let two segments of circles *pbq*, *par* be described, to contain angles respectively equal to CBD and CAE: Then draw *pa* (*by Prob.* 27.) so that the part thereof *ba*, intercepted by the two peripheries, shall be equal to



BA; and, having joined *b*, *q*, and *a*, *r*, make BP = *bp*, BQ = *bq*, AR = *ar*, and let PQ, PR, and QR be drawn for the sides of the triangle required.

DEMONSTRATION.

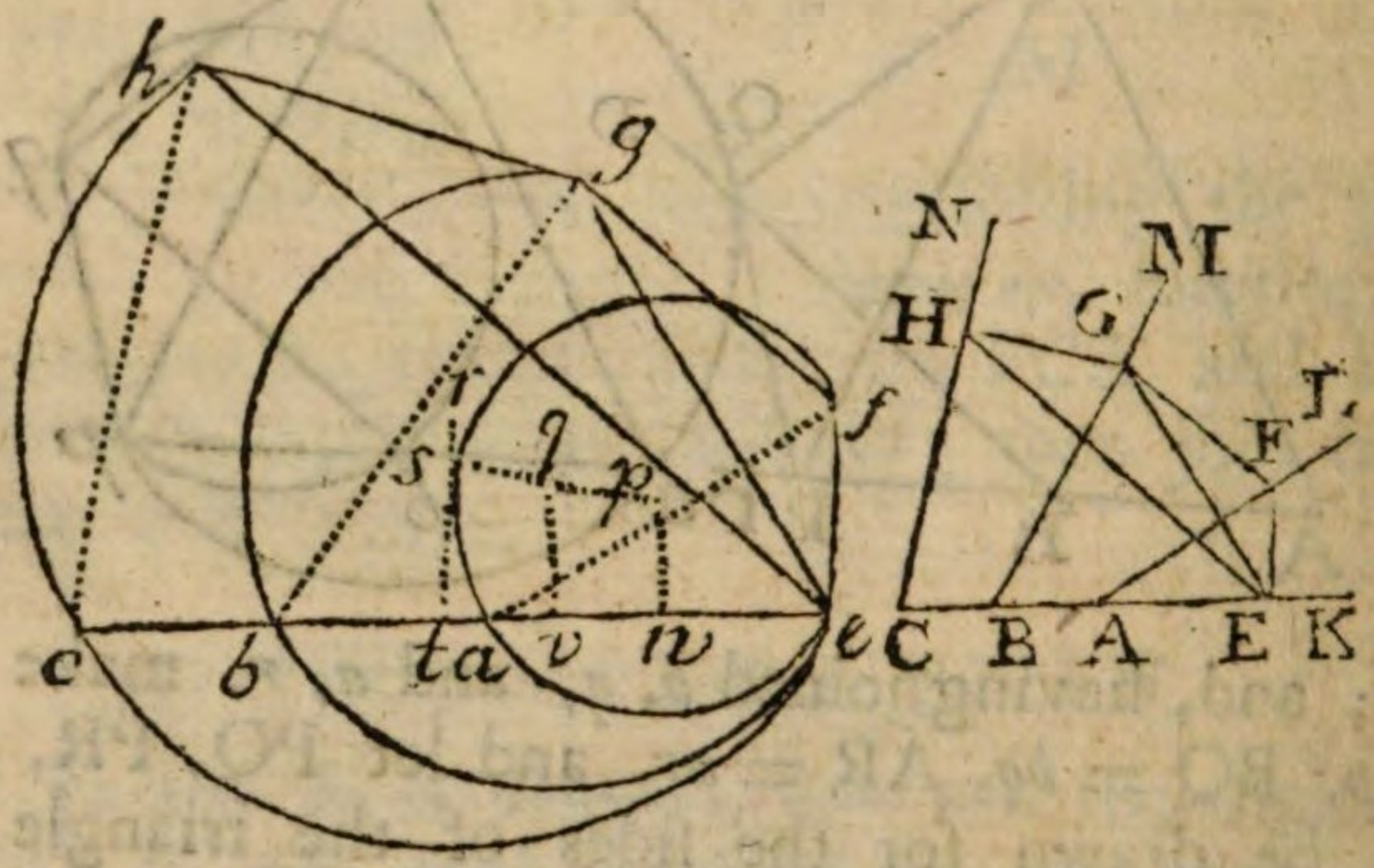
The triangles PAR, *par*; PBQ, *pbq*, being equal and alike in all respects (*by Constr.*), not only the sides PR, *pr*; PQ, *pq*, but the angles QPR, *qpr*, will be equal; and, consequently, the two triangles PQR, *pqr* also equal and like to each other. *Q. E. D.*

PROBLEM XLI.

To deſcribe a trapezium ſimilar to a given one efgh, *having its angular points placed in four right-lines* CN, BM, AL, CBAK *given by poſition.*

CONSTRUCTION.

Let three points *p*, *q*, *r* be found, from whence, as centers, ſegments of circles may be deſcribed, upon *ef*, *eg*, *eh*, to contain angles equal to the three given angles KAL, KBM, KCN, reſpectively; draw *pq*, in which, produced (if neceſſary) take *qs* to *qp* in the proportion of BC to AB;



and, having drawn *rst*, make *ec* perpendicular thereto, interſecting the three circles in *a*, *b*, *c*. Take AE : AB : : *ae* : *ab*; and make the angles CEH, CEG, and CEF, reſpectively equal to *ceh*, *ceg*, and *cef*; then let H, G and G, F be joined; and, I ſay, the trapezium EFGH will be ſimilar to the given one *efgh*.

DE-

DEMONSTRATION.

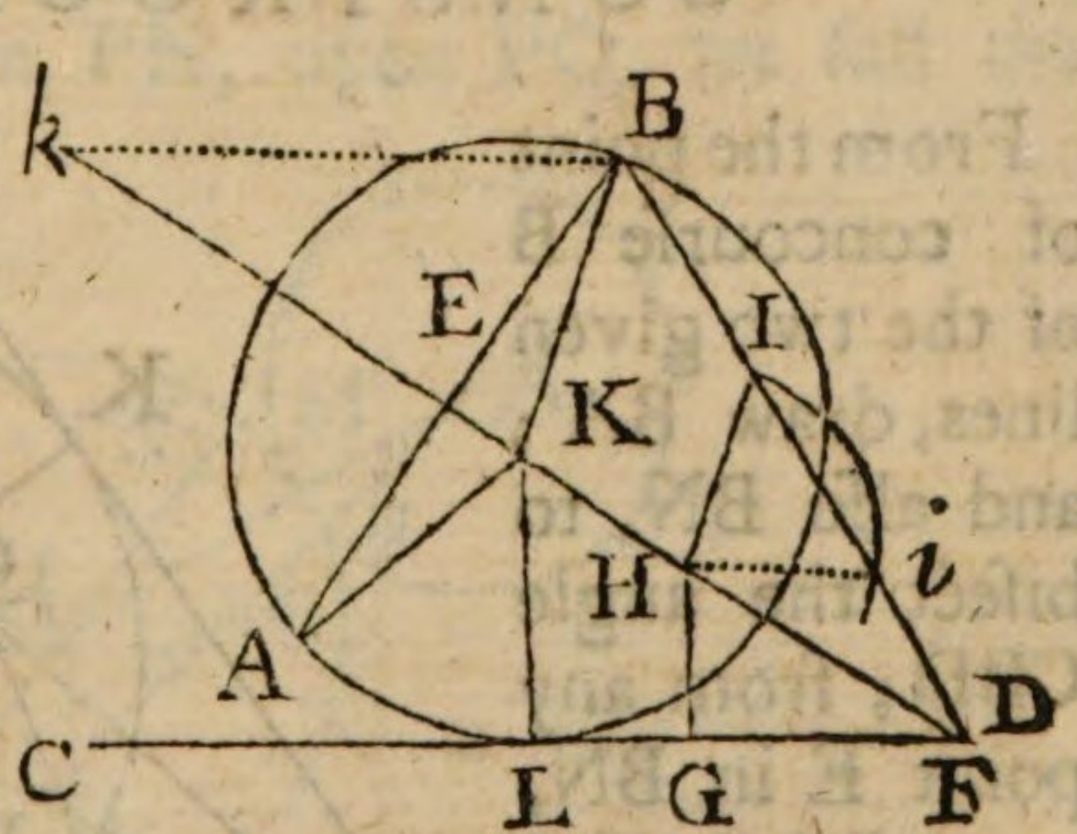
Let *af*, *bg*, *ch*, be drawn, and alſo *pw*, *qv*, perpendicular to *ec*. So ſhall $ab = eb - ea = 2ev - 2ew = 2vw$; and $bc = ec - eb = 2et - 2ev = 2vt$: And therefore *ab* : *bc* : : *vw* : *vt* : : *pq* : *qs* (*by* 13. 4.) : : AB : BC (*by Conſtr.*). Again, ſeeing the triangles EAF, *eaf*; EBG, *ebg*; ECH, *ech* are, reſpectively, equiangular (*by Conſtr.*) it will be EG : *eg* : : EB : *eb* : : AE : *ae* (*by Conſtr. and Compoſition*) : : EF : *ef* (*by* 14 4); and ſo the triangles GEF, *gef* (having the ſides about the equal angles proportional) are ſimilar. And, in the very ſame manner, the two remaining triangles EGH, *egh* (and conſequently the whole trapeziums EFGH, *efgh*) will appear to be ſimilar. *Q. E. D.*

PROBLEM XLII.

To deſcribe the circumference of a circle through two given points A, B, *which ſhall touch a right-line* CD *given by poſition.*

CONSTRUCTION.

Draw AB, which biſect in E by the perpendicular EF, meeting CD in F; from any point H in EF, draw HG perpendicular to CD; and, having drawn BF, to the ſame apply HI = HG, and parallel thereto draw BK, meeting EF in K; then from the center K, with the radius BK, let a circle be deſcribed, *and the thing is done.*



DEMONSTRATION.

Join K, A, and draw KL perpendicular to CD: Then, becauſe of the parallel lines, HG : HI :: KL : KB (*by* 21. 4.); whence, as HG and HI are equal, KL and KB are equal likewiſe: But it is evident, from the Conſtruction, that KA is = KB; therefore KB = KL = KA. *Q. E. D.*

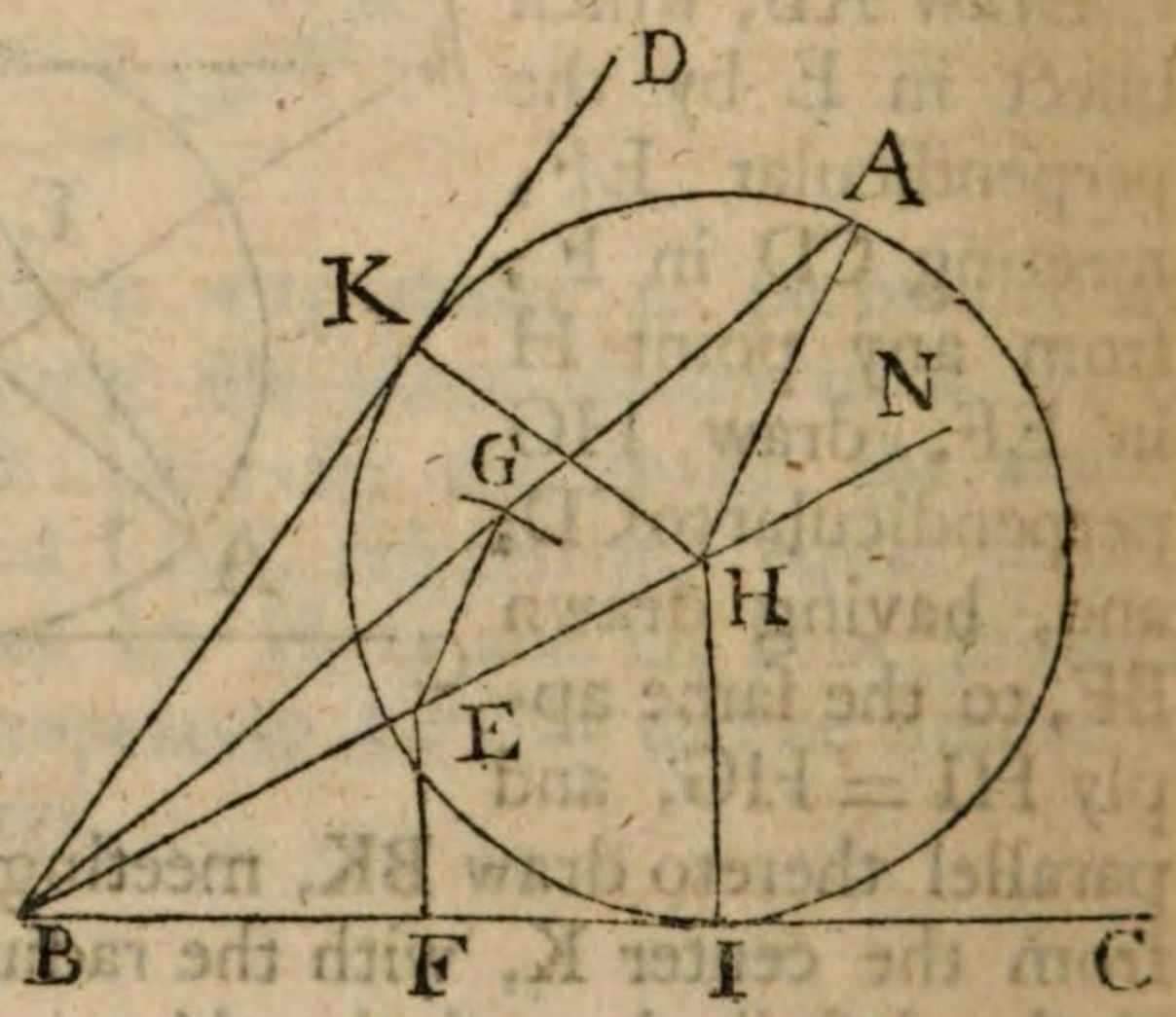
Becauſe two equal lines HI, H*i* may be applied from H to BF (except, when DIH is a right-angle) the Problem will therefore admit of two ſolutions. But when a perpendicular let fall from H upon BF is leſs than HG; the Problem will be impoſſible. And the like is to be underſtood in the conſtruction of the ſubſequent Problems.

PROBLEM XLIII.

To deſcribe the circumference of a circle through a given point A, *which ſhall touch two right-lines* BC, BD *given by poſition.*

CONSTRUCTION.

From the point of concourſe B of the two given lines, draw BA; and alſo BN, to biſect the angle CBD; from any point E in BN, upon BC, let fall the perpendicular EF, and to BA apply EG =



EF,

EF, parallel to which draw AH, meeting BN in H; then from the center H, with the interval HA, let a circle be described, *and the thing is done.*

DEMONSTRATION.

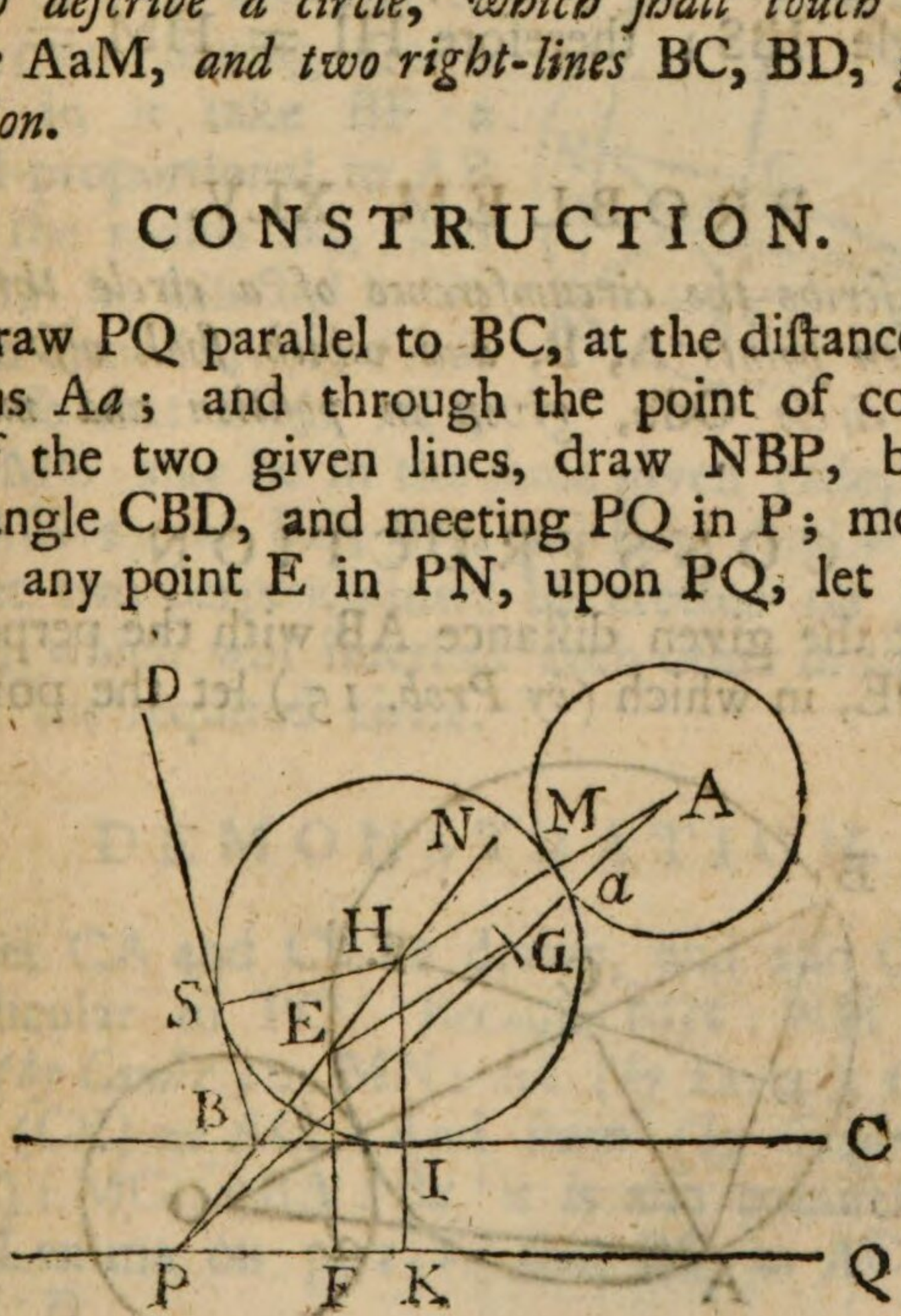
Upon BC and BD let fall the perpendiculars HI and HK; which are manifestly equal, because *(by Constr.)* the angle HBI = HBK: Moreover, as EF and EG are equal, HI and HA are also equal (*by* 21. 4). *Q. E. D.*

PROBLEM XLIV.

To describe a circle, which shall touch a given circle AaM, *and two right-lines* BC, BD, *given by position.*

CONSTRUCTION.

Draw PQ parallel to BC, at the distance of the radius A*a*; and through the point of concourse B of the two given lines, draw NBP, bisecting the angle CBD, and meeting PQ in P; moreover, from any point E in PN, upon PQ, let fall the



perpen-

perpendicular EF, and from the ſame point, to PA, apply EG = EF; draw AH parallel to EG, meeting the periphery of the given circle in M, and the right-line PN in H, from which laſt point, as a center, through M, deſcribe the circumference of a circle; *and the thing is done.*

DEMONSTRATION.

Draw HS perpendicular to BD, and HK to PQ, interſecting BC in I.

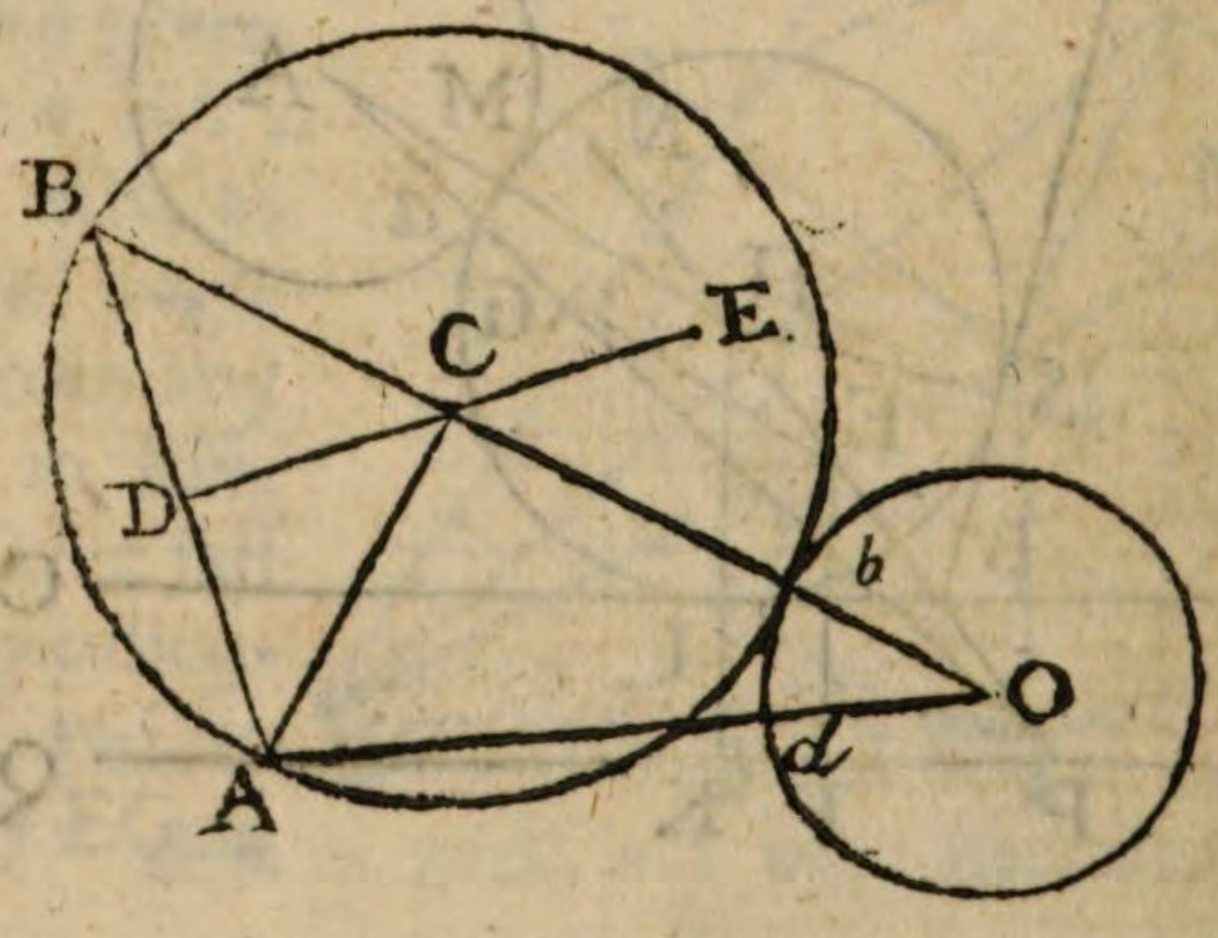
Then, becauſe EG and EF are equal *(by Conſtr.)*, HA and HK *(by* 21. 4.) are likewiſe equal; from which take away KI = AM (A*a*), and the remainders HI, HM will be equal: But, it is evident, that HI is = HS, becauſe BH biſects the angle IBS; therefore HI = HM = HS. *Q. E. D.*

PROBLEM XLV.

To deſcribe the circumference of a circle through two given points A, B, *and which ſhall alſo touch another circle* Odb, *given in poſition and magnitude.*

CONSTRUCTION.

Biſect the given diſtance AB with the perpendicular DE, in which *(by Prob.* 15.) let the point C



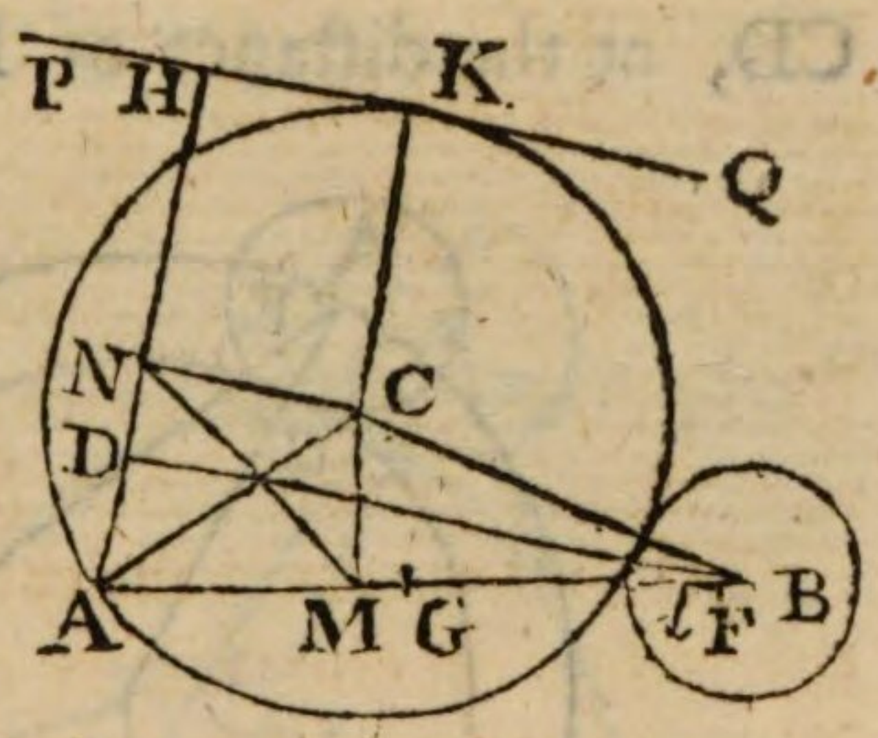
be

be ſo determined, that CO (when drawn) ſhall exceed CA by the radius O*d* of the given circle. Then that point, it is manifeſt, will be the center of the circle to be deſcribed.

PROBLEM XLVI.

Through a given point A, *to deſcribe the circumference of a circle, which ſhall touch a given circle* B, *and alſo a right-line* PQ *given by poſition.*

CONSTRUCTION.



Make AH perpendicular to PQ, and BD to AH; and having drawn AB, in it take BF a third-proportional to AB and the radius B*d*, and let AF be biſected in G: Then draw MN (*by Prob.* 35.) ſo that MN, NH, and MG may be in the ſame given ratio, among themſelves, as BD, BA, and B*d*: And at M and N let two perpendiculars be erected on AB and AG; which will interſect each other in the center C of the required circle.

DEMONSTRATION.

Let CA and CB be drawn, and alſo CK perpendicular to PQ. Becauſe MN : NH : : BD : BA (*by Conſtr.*) : : MN : AC (*by* 22. 4.), thence is NH (CK) = AC. And ſince (*by Conſtr.*) NH (AC) : MG : : BA : B*d*, it is alſo manifeſt, from the Lemma on p. 222. that BC = AC + B*d*. *Q. E. D.*

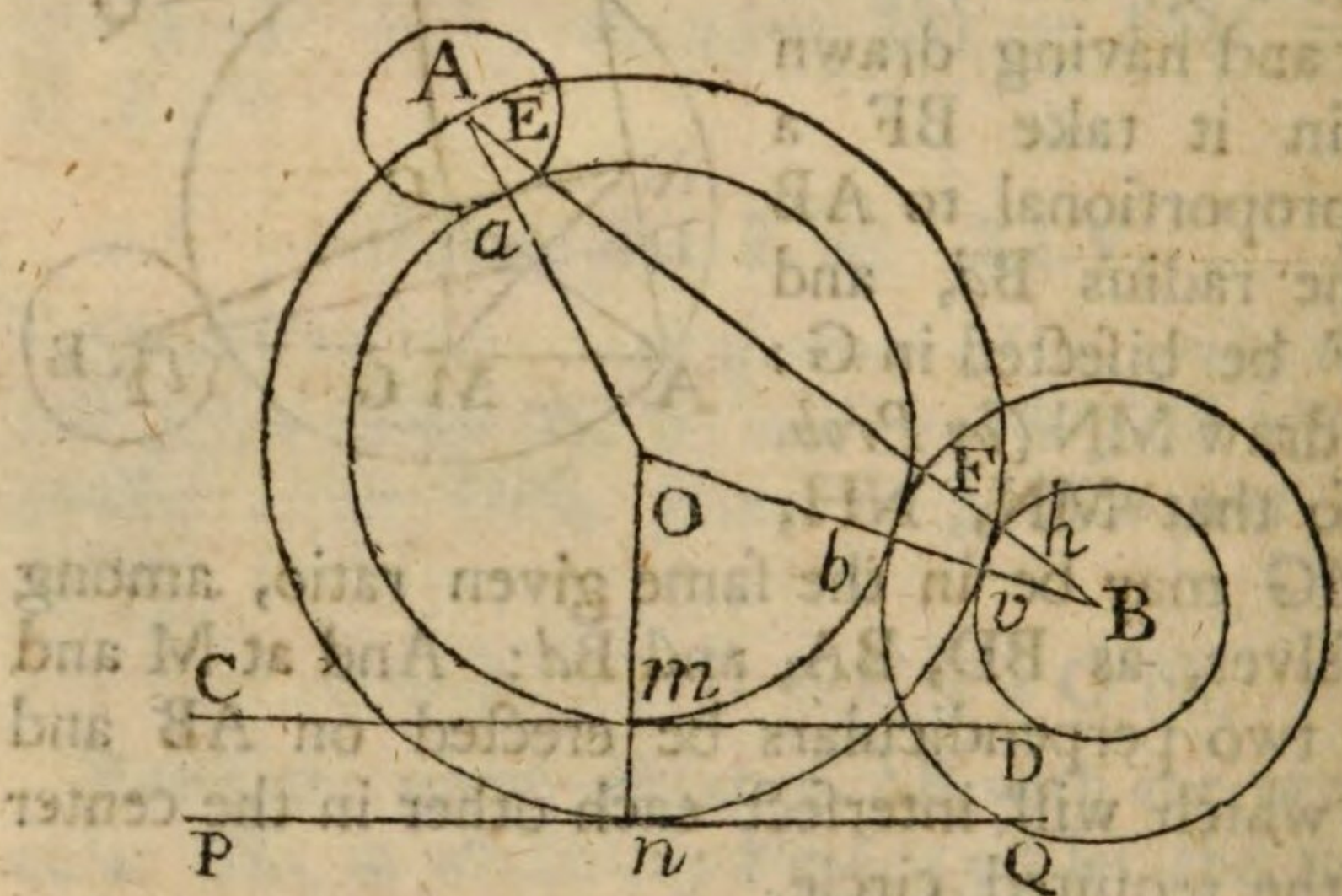
PRO-

PROBLEM XLVII.

To describe a circle O*amb, which shall touch two given circles* AEa, BFb, *and also a right-line* CD, *given by position.*

CONSTRUCTION.

From the radius BF of the greater circle, take away F*b* equal to the radius AE of the lesser; and from the center B, with the interval B*b*, describe the circle B*bv*; also draw PQ parallel to CD, at the distance of F*b* or AE: Then, by the



last Problem, let the center O of a circle be found, whose circumference shall pass thro' A, and touch PQ and B*bv*; and the same point O will, likewise, be the center of the required circle O*amb*.

DEMONSTRATION.

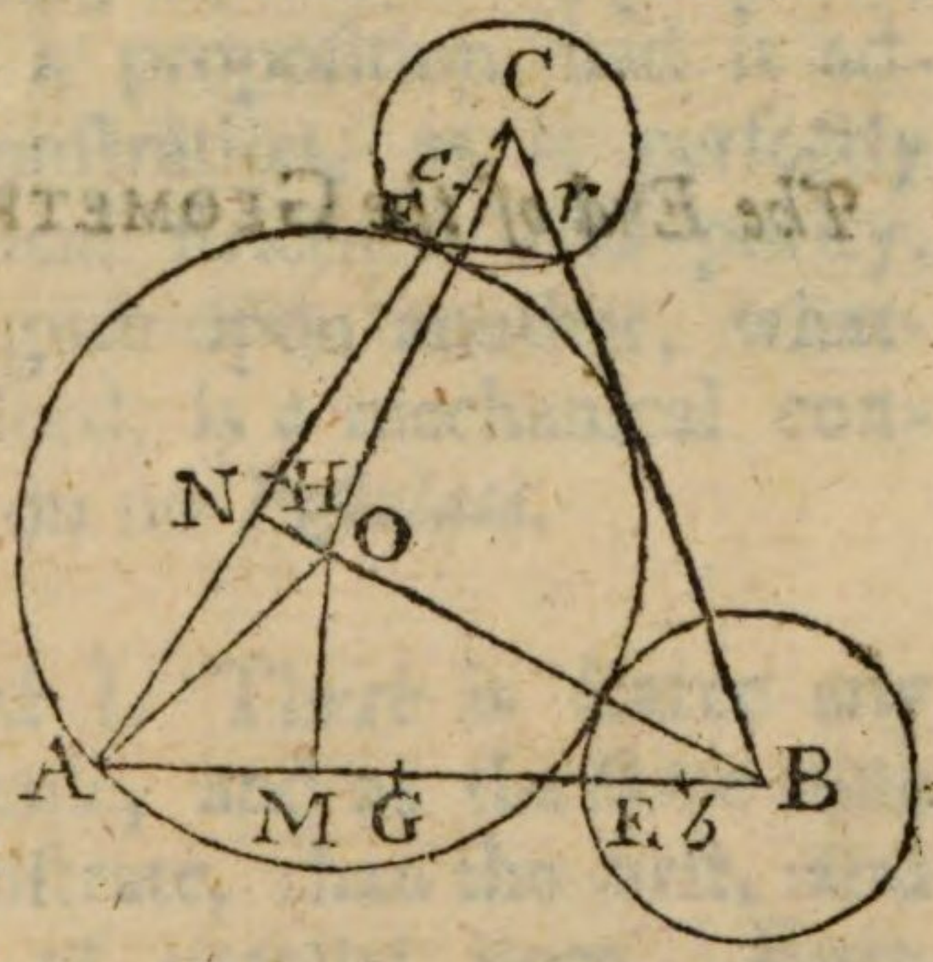
Draw O*n*, perpendicular to PQ, cutting CD in *m*; also let OA and OB be drawn intersecting the circles

circles A and B in *a* and *v*: Then, since *(by Conſtr.)* AO = *v*O = *n*O, and A*a* (AE) = *bv* (F*b*) = *nm*, let theſe laſt be reſpectively taken from the former, and there will remain O*a* = O*b* = O*m*. *Q. E. D.*

PROBLEM XLVIII.

To deſcribe the circumference of a circle through a given point A, *which ſhall touch two other circles* B *and* C, *given in poſition and magnitude.*

CONSTRUCTION.

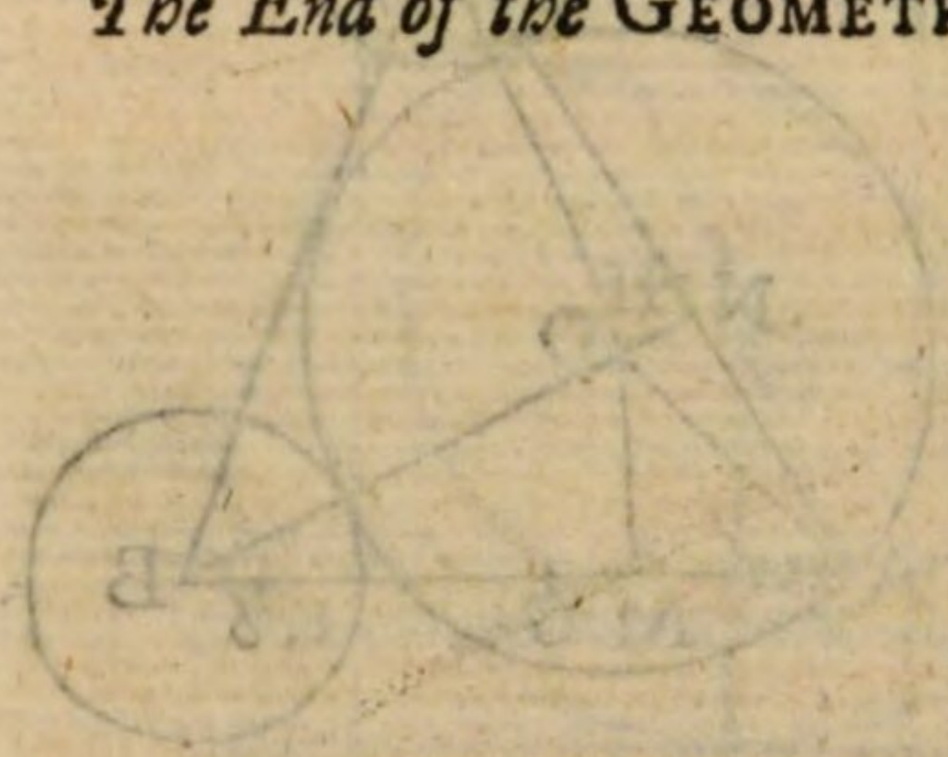


To the centers of the given circles, draw AB and AC; in which take B*b* = a third-proportional to AB and the radius BE; and C*c* = a third-proportional to AC and the radius CF: biſect A*b* and A*c* in G and H, and let F*r* be drawn parallel to AB, meeting BC in *r*. Then *(by Prob.* 34.) draw AO, ſo that OM and ON being drawn perpendicular to AB and AC, the three lines AO, GM and HN ſhall have the ſame given ratio among themſelves, as AB, BE and F*r*. Then ſhall the point O be the center of the required circle: For, ſince *(by Conſtr.)* AO : GM : : AB : BE; and AO : HN (: : AB : F*r*) : : AC : CF; it is manifeſt, from the Lemma on p. 222. that OB = AO + BE; and OC = AO + CF. *Q. E. D.*

In

In the very ſame manner, a circle may be deſcribed to touch three given circles; the Problem amounting to no more than, *To find a point from whence lines, drawn to three given points, ſhall have given differences:* Since a point ſo found will always be the center of the required circle, as well when the three given circles are to touch that circle *inwardly*, as when they are all required to touch it *outwardly*.

The End of the GEOMETRICAL CONSTRUCTIONS.



NOTES

NOTES

GEOMETRICAL and CRITICAL

ON THE

Elementary Part of this WORK.

AXIOM 10. *Book* I. What is here laid down, as an Axiom, would, more properly, have been made a propoſition, had it admitted of ſuch a demonſtration, as is perfectly conſiſtent with geometrical ſtrictneſs and purity. But the laying of one figure upon another, whatever evidence it may afford, is a mechanical conſideration, and depends on no *Poſtulate*.

Theor. 4. *and* 5. *Book* I. There is ſcarce any thing more obvious to ſenſe, and at the ſame time more difficult to demonſtrate, than the firſt, and moſt ſimple properties of parallel lines. Even when we have (in Theor. 4.) proved the poſſibility of the exiſtence of ſuch lines, we cannot from thence infer, that their diſtance from one another is every-where the ſame; without having recourſe to an Axiom, which, though very evident to ſenſe, cannot be demonſtrated. Theſe difficulties wholly ariſe from our not having any properties, previouſly demonſtrated, whereby the progreſs of a right-line, produced out, can be traced, with reſpect to its diſtance from ſome other right-line aſſigned;

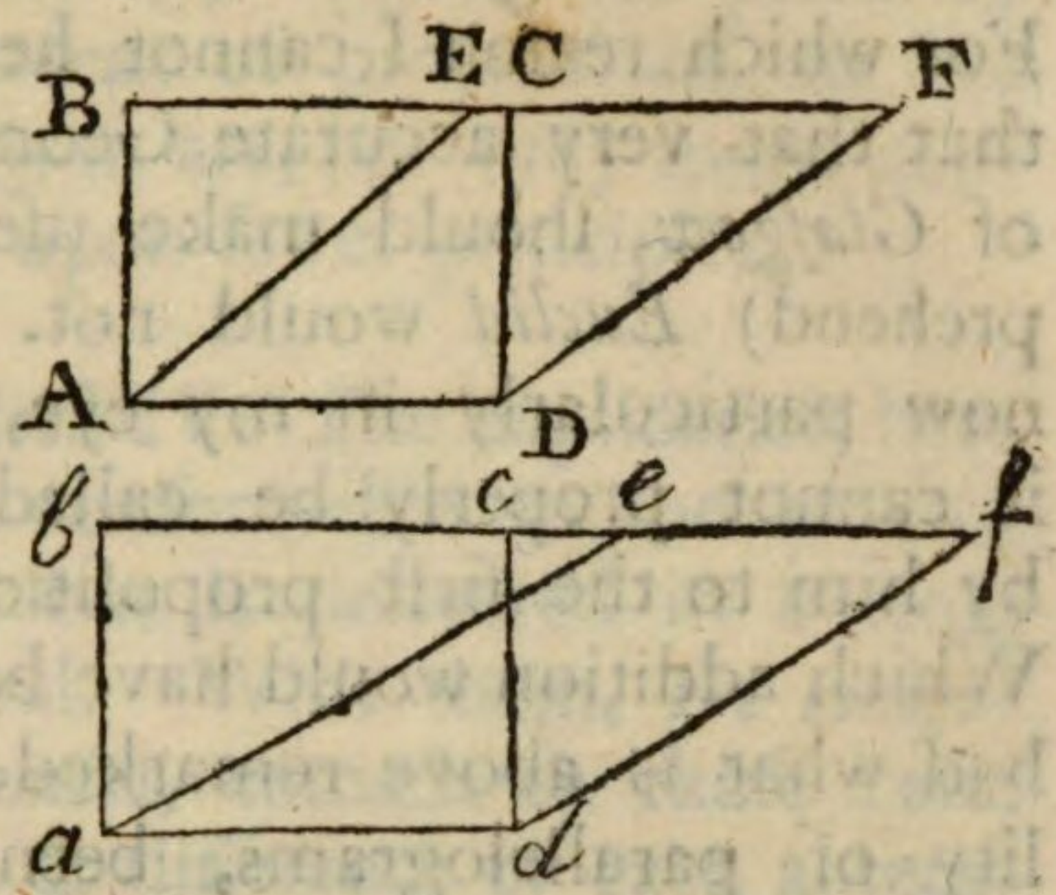
assigned; nothing less being required here, than the proportionality of the sides of equiangular triangles; which is not proved before the middle of the Fourth Book, and which depends upon these very principles.

Schol. to Theor. 5. *Book* I. As there are several conditions requisite to make up the definitions of a rectangle and square, it was necessary to shew here, that the several properties ascribed to those figures, are not incompatible one with another. *Euclid* is very strict in this particular, and never undertakes to demonstrate any thing relating to a figure, till he has proved the possibility of the existence of such figure by an actual construction.

Theor. 22. *Book* I. This proposition, which is not in *Euclid*, is of considerable use, being often wanted in determining the *Maxima* and *Minima*, in mathematical enquiries.

Theor. 27. *Book* I. This Theorem, though not in *Euclid*, is also very useful, at least, to our design: by it we are not only enabled to divide a right-line into any number of equal parts, without the help of proportions, but also to demonstrate that most important proposition, *That the homologous sides of equiangular triangles are proportional.* It is true, the method pursued here, is not exactly conformable to the idea of proportions delivered in the 6th Def. of *Euclid*'s 5th Book. But, even in that light, the demonstration will be equally easy, without inferring it from the proportionality of triangular spaces.

Theor. 1. *Book* II. This propoſition, which is not mentioned by *Euclid*, may be thought unneceſſary; but it muſt either be demonſtrated, or aſſumed, as it is wanted in almoſt every propoſition of the ſecond book: By means of it, we alſo arrive at a general, and very eaſy demonſtration of that important Theorem, *That all parallelograms* (AEFD, aefd) *which ſtand upon equal baſes, and have equal altitudes, are themſelves equal:* For, when it is known that theſe parallelograms AEFD, *aefd* are equal, reſpectively, to rectangles ABCD, *abcd*, of the ſame baſe and altitude (which is proved in Prop. 2.) it is alſo manifeſt that they muſt be equal to one another, as their equal rectangles ABCD, *abcd* are ſhewn (*by Theor.* 1.) to be equal, the one to the other. This determination is more general than that given by *Euclid*, in the 36th Prop. of his firſt book; where he demonſtrates the equality of parallelograms, whoſe equal baſes are in the ſame ſtraight line: Which may, perhaps, be thought ſufficient for the whole; becauſe, if the baſes are not in the ſame right-line, one of the two figures may be conceived to be removed, and ſo placed, that its baſe ſhall be in the ſame right-line with the baſe of the other.—But, that theſe were not *Euclid*'s ſentiments, is evident from this; He hints at no ſuch thing: And had he approved of this ſort of demonſtration, his 36th Prop. would have been intirely uſeleſs; as nothing more (after the 35th) would be neceſſary, in order to a general demonſtration, than barely to place one baſe upon the



other. But it is certain, that this is a kind of demonſtration, which *Euclid* never has recourſe to, when the thing in hand can be done without it. For which reaſon I cannot help wondering a little, that that very accurate Geometer, *Profeſſor Simſon* of *Glaſgow*, ſhould make uſe of *it*, where (I apprehend) *Euclid* would not. The place, I have now particularly in my eye, is the addition (for it cannot properly be called a corollary) made by him to the firſt propoſition of the ſixth book: Which addition would have been quite unneceſſary, had what is above remarked, reſpecting the equality of parallelograms, been fully eſtabliſhed in the ſecond book: For then the demonſtration, that parallelograms, having equal altitudes, are as their baſes, would nothing differ from that whereby parallelograms, ſtanding between the ſame parallels, are proved to be in proportion as their baſes.

Theor. 9, 11, 12, 13. *Book* II. Theſe four Theorems, tho' not in *Euclid*, are of very conſiderable uſe, particularly the two firſt of them.

Theor. 1. *Book* III. This eaſy propoſition is added in order to give the learner a proper, and more preciſe idea of the quantity of an angle, and of its diviſion in practical uſes.

Theor. 2, 3, 4, 5, 6, 7, 8. *Book* III. Theſe ſeven propoſitions comprehend all that is moſt material in the firſt ſeventeen Theorems of *Euclid*'s 3d book.—As there is no where, in this author, ſo long a run of propoſitions together that are leſs entertaining to learners, or of leſs real importance, than the greater part of the ſaid Theorems; I thought it would be of uſe to reduce the ſubſtance of them into a leſs compaſs: And I flatter myſelf, that I have not ſucceeded ill, in this particular.

Theor.

Theor. 24, 25, 26, 27, 28. *Book* III. These Theorems, which are all of very considerable use, will not, I flatter myself, appear less plain by being proved independent of proportions, as the demonstrations here given are, not only more concise, but depend also upon fewer principles.

Def. 12. *and* 13. *Book* IV. The explication here given, is not strictly conformable to the idea of ancient *Commentators*, but is delivered in a sense somewhat more general. With them, *the composition and division of ratios*, extends to those cases only, where the sum, or difference of each antecedent and its consequent, is compared with the consequent. When the antecedent is compared with its excess above the consequent, this they call the *conversion of ratios*. But in such cases where the antecedent is less than the consequent, and where the sum, or difference of the antecedent and consequent is required to be compared with the antecedent; it does not appear that any *terms* have been given to signify such a comparison. *Professor Simson* thinks, that the definitions we have, are not *Euclid*'s, but an addition by *Theon*; which to me appears highly probable: This at least seems clear, that these definitions ought, either, to have been extended to a greater number, or else to have been rendered more general. For this reason I have, after the example of modern Geometers, given the signification of those terms, so as to include all the several cases: And this, I thought, might be done with the greater propriety, as the truth of whatever is here understood, depends upon the same demonstration.

Theor. 20, 21, 22, 23. *Book* IV. All these Theorems, tho' not in *Euclid*, are of considerable use;

by the two former, the demonſtration of ſeveral others is rendered more eaſy; and the latter have been applied, to good purpoſe, in the analytical determination of ſome difficult geometrical problems.

Prob. 3, 4, 5, 6. *Book* V. In the demonſtrations here given, it may be thought, that I have affected an unneceſſary exactneſs, by making them depend on *Axioms* alone. But I was willing that theſe fundamental propoſitions ſhould have the ſame foundation and evidence, as they appear to ſtand upon in *Euclid*; without referring to any thing derived in virtue of the 4th Poſtulate. But, whether I judged well, or ill, in this particular, is of little conſequence, as the demonſtrations here given, are not leſs plain, and but very little longer, than they otherwiſe would have been.—The Problems themſelves might, indeed, have been given along with the Theorems, as they became neceſſary, according to the method purſued by *Euclid*; whereby any objection, of this ſort, might have been avoided. But, beſides ſome ſmall convenience to the learner, there is a real advantage in having the Problems all together, after the Theorems; ſince, from a great choice of properties, ready demonſtrated to our hands, we are often able to arrive at a ſhorter and better conſtruction, than could poſſibly be given from ſuch properties alone as are antecedent to *Euclid*'s ſolution of the ſame Problem; his method of writing having obliged him to introduce the leading Problems as ſoon as poſſible, in order to evince and eſtabliſh the conſiſtence of his definitions, and to open his way in a regular manner to the many uſeful Theorems thereon depending. And it is for this reaſon alone, that many of his Conſtructions are not ſo well adapted to practice, as thoſe in common uſe.

Upon

Upon which account, ſome have been precipitate enough to blame him; not ſeeing, or conſidering, that ſuch Conſtructions, tho' not ſuited to anſwer every purpoſe, were the moſt proper for his plan, and the beſt that could be given in the places where they ſtand.

Prob. 8. *Book* V. The reaſoning in this propoſition, to prove that the two circles will cut each other, may, to ſome, appear needleſs. *Profeſſor Simſon* (at p. 359. of his *Euclid)* has been a little ſevere upon me, on this head, for attempting to ſupply, what I thought, a ſmall defect in *Euclid*. "Who is ſo dull (ſays he) tho' only beginning to "learn the Elements, as not to perceive that the "circle deſcribed from the center F, *&c.*" It is not without a real concern that I here ſee this able Geometer drop his own character ſo far, as to expreſs himſelf in a manner ſo very *ungeometrical*. If the thing is, indeed, eaſy to be perceived, it muſt be ſo, either, as an immediate object of the ſenſes, that is, in plain terms, by inſpection; or elſe it muſt be in conſequence of geometrical reaſonings antecedent to the thing itſelf. Now I am clear that he would not be thought to mean the former; and, as to the latter, nothing had been given from whence the evidence of the inference could be ſo clearly ſeen: For, tho' it is proved, that any two ſides of every triangle are greater than the third ſide, it would be abſurd to urge that concluſion in the caſe before us; becauſe the queſtion here, is, whether a triangle, under certain ſpecified conditions, can, or cannot be formed? and, therefore, to conclude any thing from the properties of triangles, would be ridiculous, and nothing leſs than begging the queſtion. That the determination propoſed limits the Problem, no body will diſpute: But then, is it not neceſſary that

this ſhould be proved, in Elements of Geometry, where a reaſon for every thing is, or ought to be given? From this conſideration, I cannot intirely approve of the emendation propoſed by this Editor to *Euclid*'s 24th Prop. Book I. For, tho' the addition there made, does indeed reſtrain the propoſition to one caſe; yet this ought to have been demonſtrated, by ſhewing that the point F (vid. p. 29.) muſt in conſequence of ſuch reſtriction, neceſſarily fall below the line EG; but this is not done. Many other inſtances might be produced to ſhew, that this gentleman, who often appears a little too haſty and ſevere in his cenſures, is not, himſelf, every-where equally guarded. In Prop. I. Book III. he bids you to draw a ſtraight-line within a circle, without ſpecifying that it muſt terminate in the circumference; and, what is a great deal worſe, he here very improperly uſes the word *within*; when the propoſition itſelf is laid down in order to prove, in the ſubſequent one, that ſuch line muſt neceſſarily fall *within* the circle. Theſe are, it is true, but little matters; but leſs than theſe have fallen under this gentleman's notice. At p. 358. M. *Clairaut* is glanced at, for an inadvertency of this ſort. And, in the note at the head of p. 415. it is ſaid, "The words, *for a ſtraight-line cannot* "*meet a ſtraight-line in more than one point*, are left "out, as an addition by ſome unſkilful hand; for "this is to be demonſtrated, not aſſumed." Now can it poſſibly ſhew any want of ſkill in an editor, to refer to an Axiom which *Euclid* himſelf had laid down (Book I. N° 14.) and not to have demonſtrated, what no man can demonſtrate?

Prob. 16, 17, 18. *Book* V. Theſe three Problems, tho' not ſo frequently wanted as ſome of the preceding ones, are nevertheleſs of very conſiderable uſe. The two laſt of them are the ſame, in

in effect, with the 28th and 29th of *Euclid*'s 6th book; but are here put down in a manner rather more commodious.

Axiom, *p.* 131. This Axiom is ſubſtituted inſtead of the common definition of equal ſolids, which, I really think, is too bad to be the work of *Euclid*. It is not a definition, but a propoſition, "whoſe truth or falſhood ought (as a very judi-"cious writer obſerves) to be demonſtrated, not "aſſumed." Neither is it at all conformable to *Euclid*'s manner of writing, where he eſtabliſhes the foundation whereon the equality of plane figures is grounded; which he does, not by means of a definition, but from the application of two of the moſt ſimple figures to each other; ſo that, from the coincidence of their bounds, their equality may appear manifeſt. And this method we have purſued in treating of ſolids; without which a clear and diſtinct idea of their equality can be no more obtained, than of the equality of plane figures independent of the 4th Prop. of the firſt book, which is our 10th Axiom. I ſhould have ſaid a good deal more on this head, but I find that *Profeſſor Simſon* has already placed this matter in ſo ſtrong and clear a light, as to render any farther apology, or comment unneceſſary here. Tho' I muſt confeſs, that, had this gentleman's work come into my hands * before the elementary part of my own had

* *This did not happen till the middle of laſt November; when being in town in company with my bookſeller, and being preſſed by him to finiſh; I acquainted him that every thing was actually ready, except the Preface, which would coſt me ſome pains, ſince it would be neceſſary to obviate ſome objections, particularly with regard to the reaſons for my rejecting* Euclid's *definition of equal ſolids, and building upon a different foundation. On which, he immediately let me know, that Mr.* Rob. Simſon *had already cleared*

had been intirely printed off, the definition of ſimilar ſolids, which I have given, from *Euclid*, would have been delivered under a form ſomewhat different. For, tho' it involves no abſurdity, as it now ſtands, yet there are certain caſes (but ſuch indeed as do not occur in any Elements of Geometry) where it will not afford the preciſe idea it ought to convey.

Poſtulatum, *p.* 131. The unſatisfactory and inconcluſive demonſtration given to *Euclid*'s 2d Prop. Book XI. (by *Euclid* himſelf, or ſome leſs ſkilful *Editor*) ſeemed to render ſomething of this ſort neceſſary. In that demonſtration it is taken for granted, that one part of the triangle, at leaſt, muſt be in the ſame plane. But it has been very juſtly obſerved, that a curve-ſurface may be bounded by three right-lines: Nor does it ſeem eaſy to form a clear idea, that even a part of any one of the three lines will be in the ſame plane with one of the others, unleſs by conceiving a plane to be turned about upon the one, till it meets with, or falls upon ſome point in the other. And I have the ſatisfaction in this particular, to ſee my ſentiments exactly agree with thoſe of a very good judge, whoſe name, I have, more than once, had occaſion to mention in theſe notes. It is true, he makes that a Theorem, which I lay down as a Poſtulate. But, ſince a plane can no more be turned about upon a line, than a line can be drawn from one point to another, it ſeemed to me, that the one

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up that point; and expreſſed his ſurprize that a work of ſo much repute, wherein (he told me) my own name was more than once mentioned, had not come into my hands. A copy of which I received from him the next morning. In conſequence whereof I changed my firſt deſign of writing a long Preface; thinking it would be better to give what I had to offer, in notes, after the example of this Editor.

was

was as properly a Poſtulate, as the other. However, whether this be, or be not allowed, is immaterial, as the degree of evidence is preciſely the ſame. There is, indeed, one reaſon, why this Theorem, or Poſtulate, ought to have preceded that gentleman's demonſtration of Prop. 1. Book XI; It is there wanted: For, in the Corollary on which that demonſtration is made to depend, the lines AB, BD, BC are ſuppoſed to be all in the ſame plane; which ought by no means to be aſſumed in the 1ſt of the 11th. *Euclid*'s 10th Axiom, which that Corollary is intended to ſupply, and by which the propoſition is uſually demonſtrated, is not limited by any ſuch reſtriction.

Theor. 12. *Book* VII. This propoſition is added on account of its uſe, being the foundation on which the whole *art of perſpective* in a manner depends.

Theor. 25. *Book* VII. From this Theorem, which is very extenſive in its application, ſeveral others of conſiderable note may be deduced: one, or two of which, for the ſake of the learner, I ſhall here derive, and put down by way of example.

Let A, B, C, D denote four lines in continual proportion.

Then, ſince $\left\{\begin{array}{l} A : B :: C : D, \\ A : B :: B : C, \\ A : B :: A : B, \end{array}\right\}$ it follows (*from Theor.* 25.) that

$A^3 : B^3 :: CBA : CBD :: A : D$ (*by* 22. 7.) or that, of four lines in continual proportion, the cube of the firſt is to the cube of the ſecond, as the firſt line is to the fourth.

Again, let, $A : a :: B : b :: C : c$ (where A, *a*; B, *b*; C, *c*, may be ſuppoſed to repreſent the homologous ſides of two ſimilar parallelepipedons).

Then,

Then, since $\left\{\begin{array}{l} A : a : B : b \\ A : a :: C : c \\ A : a :: A : a \end{array}\right\}$ it alſo follows,

that, $A^3 : a^3 :: BCA : bca$; or, that ſimilar parallelepipedons are to one another, as the cubes of their homologous ſides.—The proportionality of ſimilar parallelepipedons, deſcribed upon proportional lines, is alſo included in the ſame Theorem; being no other than that caſe of it, where the propoſed ratios are all equal.

Theor. 5. *Book* VIII. The demonſtration of this Theorem might have been delivered under a form ſomewhat different, by aſſuming two other ſolids (without regard to figure) the one leſs, and the other greater than the propoſed parallelepipedon IP, and proving that cylinder muſt, *alſo*, be greater than the one, and leſs than the other: which is done by means of *Lem.* 1. that is, by taking P*m*, or P*t*, ſo ſmall a part of IP, as to be leſs than the difference between the given parallelepipedon and either of the ſaid ſolids: from whence the demonſtration will proceed on, in the ſame manner in which we have given it. But, as theſe additional conſiderations would have increaſed the number of ſchemes, and lengthened the proceſs, without adding one jot to the degree of evidence, it was thought proper, for the ſake of the learner, to omit them.

Theor. 8. *Book* VIII. This Theorem is not ſo uſeful as the Corollaries that follow from it, which are all of very great importance: In the 3d and 4th of them, the proportion of all kinds of priſms and pyramids is aſſigned, without the aſſiſtance of the uſual demonſtrations given for this purpoſe; which, tho' ſufficiently evident in themſelves, are often found a little perplexing to learners, on account

count of the ſchemes, wherein ſo great a number of lines is neceſſary.

Nothing, in the courſe of theſe notes, has been ſaid relative to the Theorems on proportions, tho' ſo nice and critical a ſubject, and tho' the method I have therein purſued may ſtand in need of ſome apology. But, indeed, the whole of what I have to offer on this head, was too much to be comprized in the compaſs of one ſingle note, and could not ſo properly be delivered in ſeveral detached ones: For which reaſon, I ſhall here throw together all that I have to advance on that ſubject.

There are two objections that may be brought againſt the method in which proportions are treated of in this work; the one, grounded on the impoſſibility of dividing every magnitude into equal parts; and the other, on the incommenſurability of two or more magnitudes of the ſame kind, when compared with each other. The firſt of theſe objections appears, to me, to have very little weight. For, tho' a magnitude may be ſo conſtituted, that the diviſion of it into an aſſigned number of equal parts cannot be, *actually*, effected by any geometrical conſtruction; yet it is no leſs evident, for that reaſon, that every ſuch magnitude has not really its third, fourth, or other aſſigned part, tho' we are at a loſs how to take it; or, in other words, it ſeems very clear to conceive, that in every propoſed magnitude, whatever its figure may be, a leſs magnitude is contained, which, repeated an aſſigned number of times, ſhall be equal to the magnitude given. If, as the moſt rigid judges allow, every plane figure is equal to *ſome* ſquare, and every ſolid equal to *ſome* parallelepipedon; then the parts of the ſquare, or parallelepipedon, which are actually determinable by a geometrical

metrical conſtruction, will alſo be like parts of the figure firſt propoſed, and ſuch as we conceive to be taken.

The other objection, depending on the incommenſurability of magnitudes, is a matter of real difficulty; which we have taken ſome pains to obviate, in the *Scholia* to our 3d and 7th Theorems. *Euclid*, himſelf, ſeems to have been not a little embarraſſed with *it*, if we may be allowed to judge from the different methods he has left us in his 5th and 7th books; the former whereof, which is ſuited to include the buſineſs of incommenſurables, being nothing near ſo eaſy and natural as the latter: It has, it is true, the advantage of being general; but, that the principles whereon it is grounded, are neither ſo ſimple, nor ſo evident as might be wiſhed for, the many diſputes about them, ſince *Euclid*'s time, by Geometers of the firſt rank, will in a great meaſure evince. And farther, it ſeems ſufficiently plain, from *Euclid*'s own authority, that he himſelf was not intirely pleaſed with his own performance on this head; or that he was convinced (at leaſt) that it had not *every advantage:* For, otherwiſe, it will be very difficult to account for his having demonſtrated many things in his 7th book, by another method, whoſe demonſtrations had been actually given before, in the 5th, under a different form. For theſe reaſons, when I ſee the extravagant commendations that have been laviſhed on this 5th book of *Euclid*, I am no farther convinced by them, than that great men may ſometimes launch out too far in behalf of opinions which they have adopted. And I believe that, whoever has read the notes on the 5th book, by that great reſtorer of *Euclid*, *Profeſſor Simſon*, will be apt to conclude, that thoſe high encomiums are a little miſapplied. Indeed, if all that is advanced in thoſe notes be allowed of, I think the author of them has proved

too

too much; and this ſuperb fabric of proportions, reared with ſo much art, ſtands upon a tottering foundation. It is not by choice that I go out of my way to play the *critic*; but as the writers againſt the vulgar and indiſtinct notion of proportions (as they term it) are very ſevere in their cenſures, and aſſume a great ſuperiority, from the boaſted *accuracy of their reaſonings*, it may be neceſſary to ſhew my reader, that, tho' what he is *here* taught on proportions, is liable to ſome objections, the method which ſome ſo greatly prefer, has *alſo* its difficulties; and that there are *other objections* to it beſides its *obſcurity*. And this I ſhall make appear from this learned Commentator's own authority and conceſſions; and in order thereto, ſhall firſt refer to his note on Prop. 10. which proceeds thus. "It was "neceſſary to give another demonſtration of this "propoſition, becauſe that which is in the *Greek*, "and *Latin*, or in other editions, is not legitimate. "For the words *greater*, *the ſame*, or *equal*, *leſſer*, "have a quite different meaning when applied to "magnitudes and ratios, as is plain from the 5th "and 7th definitions of Book 5. by help of theſe let "us examine the demonſtration of the 10th Prop. "which proceeds thus, *&c.*" He then goes on, in a long note, to ſhew the inſufficiency of a demonſtration, which had been received, by all, as perfectly genuine and ſatisfactory; and at laſt comes to this concluſion. "Wherefore the 10th Propo- "ſition is not ſufficiently demonſtrated. And it "ſeems, that he who has given the demonſtration "of the 10th Propoſition, as we now have it, in- "ſtead of that which *Euclid* or *Eudoxus* had given, "has been deceived in applying what is manifeſt, "when underſtood of magnitudes, unto ratios, "*viz.* that a magnitude cannot be both greater "and leſs than another. That thoſe things which "are equal to the ſame are equal to one another,

"is

" is a moſt evident Axiom when underſtood of " magnitudes, yet *Euclid* does not make uſe of it " to infer that theſe ratios which are the ſame to " the ſame ratio, are the ſame to one another; " but explicitly demonſtrates this in Prop. 11. of " Book 5. The demonſtration we have given of the " 10th Prop. is no doubt the ſame with that of " *Eudoxus* or *Euclid*, as it is immediately and di- " rectly derived from the definition of a greater " ratio, *viz.* 7th of 5."

Here the weight of the objection reſts on its not having been proved, that, of three given magnitudes A, B, C, the ratio of A to C could not, at the ſame time, be both greater and leſs than that of B to C. But, if in the demonſtration, here rejected as inſufficient, there is any real flaw, it is chargeable on the definition of a greater and leſs ratio, as the reaſoning from it, is clear, ſtrong, and perfectly ſcientific. And I would ſeriouſly aſk the Contemners of *the vulgar and confuſed notion* of proportions, if a definition, by which it cannot be known, whether the ratio of the firſt to the ſecond of four given magnitudes, may not, at the ſame time, be both greater and leſs, than that of the third to the fourth, is really calculated to afford thoſe very accurate ideas, they pretended to? This Commentator has too much penetration not to be aware of the force of this objection, which he has attempted to obviate in one particular caſe. But the new propoſition given by him, for that purpoſe, ought to have preceded the 10th, and to have been demonſtrated independent of it. This he alſo ſeems apprized of, when he ſays, that " it cannot " be eaſily demonſtrated without the 10th, as he that " tries to do it will find." But, be this as it will, I am not at all clear that his " demonſtration of " the 10th, is the ſame with that of *Eudoxus* or " *Euclid*." *Euclid* or (if you pleaſe) *Eudoxus*, does never

never (that I know of) refer to any definition, till it has been proved, either by an actual conſtruction, or by ſome demonſtration previous to that in hand, that ſuch definition involves no abſurdity, or conditions that are incompatible one with another. If, therefore, it was conceived poſſible, that the definition of a greater and leſs ratio, could involve ſo great an abſurdity, as that, *by it*, the ratio of A to C might at the ſame time be both greater and leſs than that of B to C; this point, according to the method preſcribed by *Euclid*, ought to have been cleared up, not by means of propoſitions derived in virtue of that very definition, but by others antecedent thereto, and independent thereupon. And, to me, the 8th Prop. ſeems the proper place for the doing of this, where it might be eaſily introduced, either in the Prop. itſelf, or by way of Corollary. It is there proved, that if, of three magnitudes A, B, C, the firſt A is greater than the ſecond B, then certain equimultiples of A and B may be taken ſuch, that being compared with ſome multiple of C, the multiple of A ſhall be greater, and that of B leſs than the ſaid multiple of C. Whence, by the definition of a greater ratio, the ratio of A to C is greater than that of B to C. To which might be added—And, becauſe A is greater than B, no equimultiples of A and B can poſſibly be ſo taken, that the multiple of A ſhall be equal to, or leſs than ſome multiple of C, and that of B greater than the ſame; becauſe A being greater than B, the multiple of A is ever greater than that of B, and therefore alſo greater than that of C, which is equal to, or leſs than the multiple of B. Wherefore the ratio of A to C cannot (by the definition) be leſs (as well as greater) than the ratio of B to C.

But, notwithſtanding all that has been proved on this head, either here, or by that gentleman himſelf, the ſame objection occurs again in Prop. 13.

where

where it remains in its full force. For, tho' it be allowed, that " there are ſome equimultiples of C " and E, and ſome of D and F ſuch, that the " multiple of C is greater than the multiple of D, " but the multiple of E not greater than the mul- " tiple of F;" yet it is not demonſtrated, nor in any ſort ſhewn, that other equimultiples of thoſe quantities cannot be taken ſuch, that the very contrary ſhall happen.—If the demonſtration of the 10th Prop. has been juſtly rejected by this gentleman himſelf, as inſufficient, becauſe the impoſſibility of a contrary concluſion had not been ſhewn; can it be thought that this 13th Propoſition is, at this day, ſufficiently demonſtrated, where the ſame objection occurs, and that in a much greater latitude? I have a much better opinion of this Editor's diſcernment, than to imagine, that his paſſing this matter over in ſilence, proceeded from his not being aware of the difficulty; but it ſeems to me, that his great diſlike to the *vulgar* idea of proportion (ſo often teſtified in the courſe of his notes) would not permit him to borrow any thing from thence, however evident, and though this objection, that ſtrikes deep at the very root of proportions, might by means thereof be very eaſily removed. I ſay, the very root of proportions is deeply ſtruck at in this objection; becauſe both the alternation and equality of ratios (*ex æquali ſc. diſt.*) are grounded on the ſaid 13th Prop. and which, therefore, till the objection is removed, muſt be allowed to ſtand upon an uncertain foundation.

The principle hinted at above, whereby the difficulty might be obviated, is, that if a magnitude of any kind be given, or propounded, there may (or can) be another magnitude of the ſame kind which ſhall have to it any ratio aſſigned. This aſſumption Mr. *Profeſſor* will by no means admit of

(tho'

(tho' *Euclid* himſelf, in Prop. 2. of his 12th book, has uſed it); and, in a long note on Prop. 18. is angry with *Clavius* for having recourſe to it; affirming "That the demonſtration (given by means thereof) "is of no force;" and that "the thing itſelf can-"not (as far as he can diſcern) be demonſtrated by "the preceding propoſitions, ſo far is it from de-"ſerving to be reckoned an Axiom, as *Clavius*, "after other Commentators, would have it." That the aſſumption cannot be generally demonſtrated by the preceding propoſitions (nor even by all the propoſitions in the Elements) I readily aſſent to: but then, becauſe a thing, exceedingly obvious in its own nature, cannot be demonſtrated, is it therefore leſs proper for an Axiom? I ſhould rather take the other ſide of the queſtion, and maintain that nothing ought to be made an Axiom, which can be demonſtrated. But we are not, it ſeems, allowed to have any idea of proportion but what is contained in the 6th and 8th (or, as this Author makes them, the 5th and 7th) definitions of *Euclid*'s 5th book. And, in his note on the new Prop. marked A, He is again diſpleaſed with *Clavius*, for thinking it ſufficiently evident, *from the nature of proportionals*, that if, of four proportional magnitudes, the firſt antecedent is greater than its conſequent, the ſecond antecedent will alſo be greater than its conſequent. "As if there was "(ſays he) any nature of proportionals antecedent "to that which is to be derived and underſtood "from the definition of them." Now I cannot help thinking, with *Clavius*, that there was a nature, or idea of proportion antecedent to that given in the 6th and 8th definitions of *Euclid*'s 5th book: For, that mankind, long before the time of *Euclid*, had ſome way to ſhew, or expreſs, in what degree one magnitude was greater or leſs than another, cannot be doubted: And this was

the firſt, and natural idea of proportion: And I look upon thoſe definitions, as refinements, only, on the ſimple and natural idea, in order to take in the buſineſs of incommenſurables; whereby the original notion is ſo much obſcured, that it requires ſome ſkill, even to ſee that it is at all contained in theſe definitions. I intirely agree with this gentleman, that every demonſtration ought to be ſtrictly derived from principles before eſtabliſhed: But then, whether is it more eligible, to have recourſe to an Axiom founded (as all other Axioms are) on the evidence of ſenſe and reaſon, or to an obſcure and perplexed definition, which may, for any thing that has been proved to the contrary, involve an abſurdity?

That there is ſomething very ingenious and ſubtle in the doctrine of proportions, as delivered in *Euclid*'s 5th book, cannot be denied. All that I contend for, is, that the principles on which it is built are *obſcure*, and not ſo firmly eſtabliſhed, as to authorize its partiſans to aſſume that great ſuperiority they lay claim to, in point of geometrical ſtrictneſs.

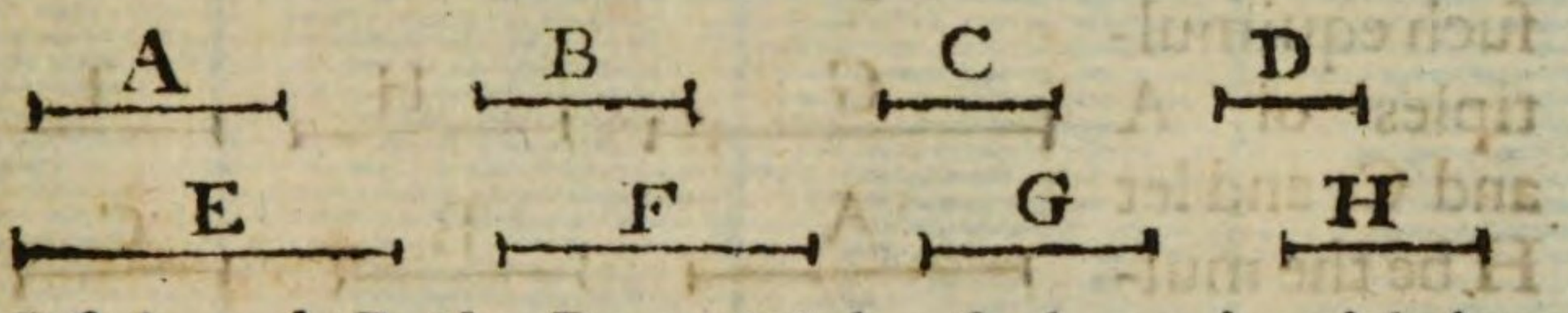
I have intimated above, that the principle is rejected, by which the conſiſtence of the definition of a greater and leſs ratio might be eſtabliſhed, without much difficulty: But I would not be thought to mean, that the ſame thing cannot poſſibly be effected any other way, becauſe I am ſatisfied that it may be done from the conſideration of multiples alone: But a demonſtration of this ſort is not eaſy.—Were I to treat of proportions from the plan laid down in the 5th book of *Euclid*, I would intirely reject the 10th and 13th propoſitions, and every thing elſe founded on the definition of a greater and leſs ratio, as being of no other uſe in the Elements, than to open the way to thoſe important Theorems on the alternation and equality of ratios; which may be better demonſtrated without them, from the definition of equal ratios alone;

which,

which, from the conditions of it, can admit of no abſurdity; and whoſe conſiſtence is evinced in Prop. 15. and ſtill more clearly in the firſt of the ſixth.

In the 14th of the 5th, whereon the alternation of ratios is grounded, it is neceſſary to demonſtrate, "That if, of four proportional magnitudes, of the "ſame kind, A, B, C, D, the firſt be greater than "the third, the ſecond ſhall be greater than the "fourth; and if equal, equal; and if leſs, leſſer." Which may be very eaſily done, independent both of the 10th and 13th, in the manner following.

Firſt, let A be greater than C.



Of A and C *(by Prop.* 8.) let ſuch equimultiples be taken, that the multiple of A ſhall be greater, and that of C leſs, than ſome multiple of B; let E and G be any two ſuch equimultiples of A and C, and F the multiple of B; ſo that E ſhall be greater than F, and G leſs than F; and let H be the ſame multiple of D, as F is of B. Therefore, becauſe E and G are equimultiples of the firſt and third, and F and H alſo equimultiples of the ſecond and fourth; and ſeeing that *(by Hyp.)* E is greater than F; it is evident, from the definition of equal ratios, that G muſt likewiſe be greater than H: Therefore much more ſhall F (which exceeds G) be greater than H; whence alſo B ſhall be greater than D *(by Ax.* 4.) B and D being like parts of F and H.

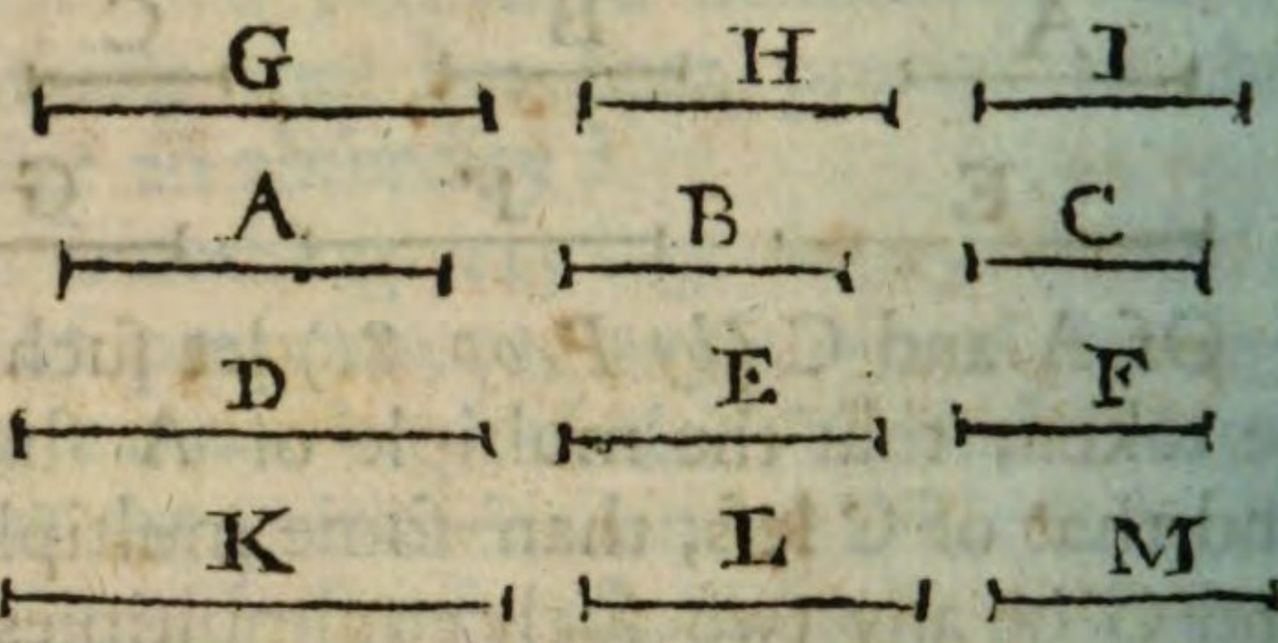
When A is leſs than C, it will be demonſtrated in the ſame manner, that B is alſo leſs than D. But when A is equal to C, no new demonſtration is neceſſary; ſince neither the 10th nor the 13th have any thing to do in this caſe.

Again, in the 20th Prop. (in which the 10th and 13th alſo enter) we are to prove, "That if there be

"three

" three magnitudes (A, B, C) and other three (D,
" E, F) which taken two and two have the ſame ra-
" tio (A : B : : D : E, B : C : : E : F) if the firſt (A)
" be greater than the third (C), the fourth (D) ſhall
" be greater than the ſixth (F); and if equal, equal;
" and if leſs, leſſer." Which may likewiſe be done without the aſſiſtance of either the 10th or the 13th, in the ſame manner, above ſpecified.

For, if A be greater than C; then of A and C (*by Prop.* 8.) ſuch equimultiples may be taken, that the multiple of A ſhall be greater, and that of C leſs, than ſome multiple of B; let G and I be two ſuch equimultiples of A and C, and let H be the multiple of B, ſo that G ſhall be greater than H, and I leſs than H; moreover take L the ſame multiple of E, as H is of B; and K and M the ſame equimultiples of D and F, as G and I are of A and C. Therefore, ſince of the four proportionals A, B, D, E, equimultiples G, K of the firſt and third, and equimultiples H, L of the ſecond and fourth, are here taken, it is manifeſt, from the definition of equal ratios, ſeeing G is greater than H (*by Hyp.*) that K muſt alſo be greater than L. And in the very ſame manner, becauſe B, C, E, F are proportionals, and H is greater than I, L will likewiſe be greater than M: Therefore much more ſhall K, which exceeds L, be greater than M. And conſequently (*by Ax.* 4.) D ſhall alſo be greater than F. —When A is leſs than C, the demonſtration is the ſame: The other caſe, when A is equal to C, does not require, nor indeed admit of any improvement.



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Elements of Geometry With Their Application to the Mensuration of Superficies and Solids, To The Determination of the Maxima and Minima of Geometrical Quantities, And To The Construction of a great Variety of Geometrical Problems

London 1760

Math.p. 529
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